150 ispitnih zadataka za vježbu podjeljenih po oblastima - detaljno raspisana rješenja ovih zadataka možete skinuti sa stranice pf.unze.ba\nabokov\za_vjezbu

Sadržaj

1 Određeni integrali. Smjena promjenjivih u određenom integralu.

1. Izračunati integrale.

$$(a)\int_2^3 3x^2\mathrm{d}x; \qquad (b)\int_0^4 (1+e^{\frac{x}{4}})\mathrm{d}x; \qquad (c)\int_{-1}^7 \frac{\mathrm{d}t}{\sqrt{3t+4}}; \qquad (d)\int_0^{\frac{\pi}{2a}} (x+3)\sin ax\mathrm{d}x.$$

2. Izračunati integrale.

$$(a)\int_0^5 \frac{x\mathrm{d}x}{\sqrt{1+3x}}; \qquad (b)\int_{\ln 2}^{\ln 3} \frac{\mathrm{d}x}{e^x-e^{-x}}; \qquad (c)\int_1^{\sqrt{3}} \frac{(x^3+1)\mathrm{d}x}{x^2\sqrt{4-x^2}}; \qquad (d)\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{2+\cos x}.$$

3. Dokazati da za parnu funkciju $f(x)$ vrijedi

$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x$$

dok za neparnu funkciju $f(x)$ vrijedi $\int_{-a}^{a} f(x)\mathrm{d}x = 0.$

2 Primjena određenog integrala

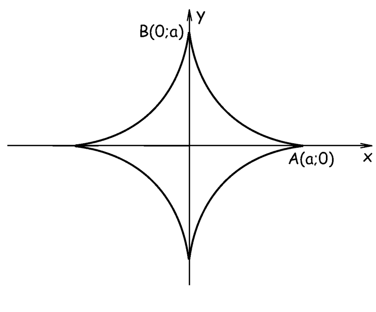
4. Izračunati dužinu luka polukubičnog paraboloida $y^2 = (x-1)^3$ između tački $A(2;-1)$ i $B(5;-8)$.

5. Izračunati dužinu luka jednog svoda cikloide $x = a(t-\sin t)$, $y = a(1-\cos t)$ (za jedan svod cikloide parametar t uzima vrijednosti od 0 do 2π).

6. Izračunati zapreminu tijela koje nastaje rotacijom krive $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ oko y-ose.

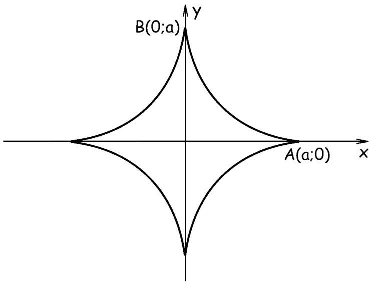
7. Figura u ravni ograničena parabolom $y = 4-x^2$ i poluravnima $y \geq 3$, $y \geq 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

8. Izračunati površinu omotača tijela koje nastaje kada dio krive $y = x^3$, koji se nalazi između pravih $x = -\frac{2}{3}$ i $x = \frac{2}{3}$, rotira oko x-ose.



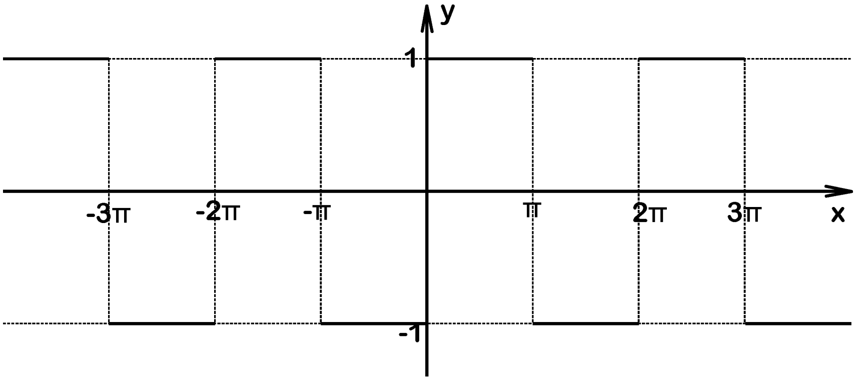
9. Izračunati površinu omotača tijela koje nastaje kada astroida $x = a\cos^3 t$, $y = a\sin^3 t$ rotira oko x-ose (grafik astroide je prikazan na slici lijevo).

10. Figura u ravni ograničena linijama $2y = x^2$ i $2x+2y-3 = 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.



11. Izračunati zapreminu tijela koje nastaje rotacijom krive $x = a\cos^3 t$, $y = a\sin^3 t$ oko x-ose (data kriva je poznata pod imenom astroida i njen grafik je prikazan na slici lijevo).

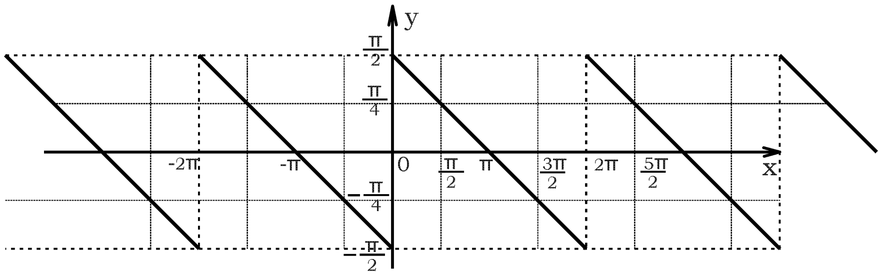
3 Furijeovi redovi



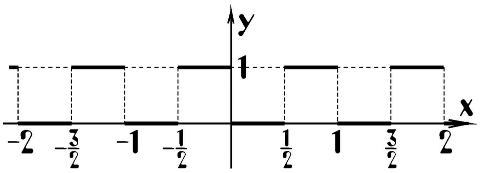
12. Funkciju definisanu grafikom pretvoriti u Furijer-ov red.

Dobijeni rezultat iskoristiti za sumiranje reda $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$.

13. Funkciju definisanu grafikom pretvoriti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(4n-1)(4n-3)}$.

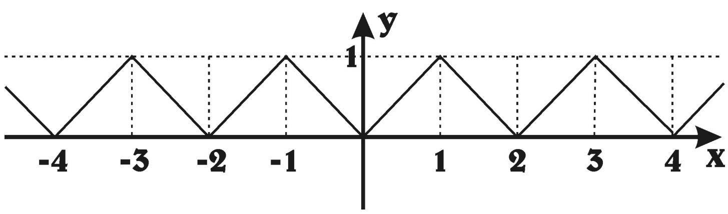


14. Pretvoriti u Fourier-ov red funkciju definisanu grafikom:

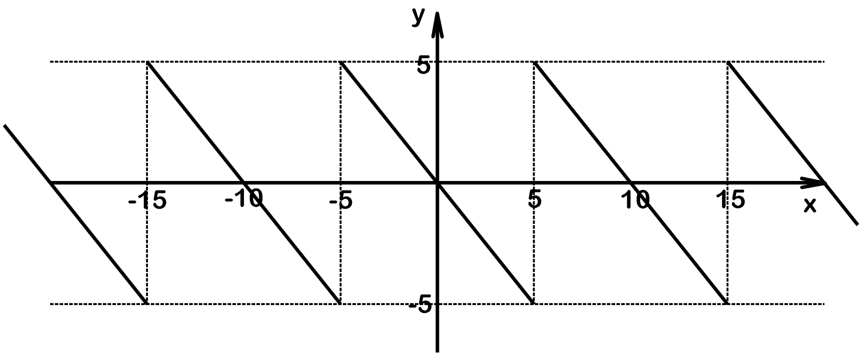


Iskoristiti dobijeni rezultat za izračunavanje sume redova $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ i $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$.

15. Funkciju definisanu grafikom pretvoriti u Fourier-ov red.



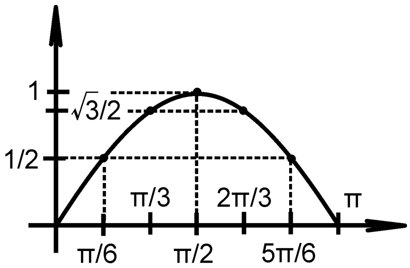
Dobijeni rezultat iskoristiti za sumiranje reda $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}$.



16. Funkciju definisanu grafikom pretvoriti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty} \frac{(-1)^n}{n} sin \frac{n\pi}{50}$.

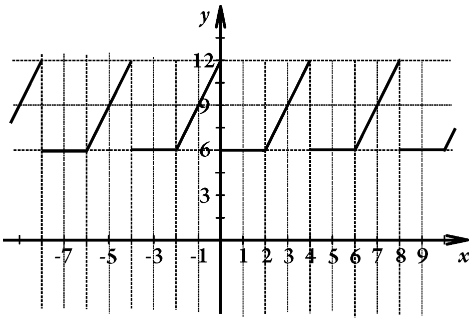
17. Razviti funkciju $f(x) = x(\frac{\pi}{2} - x)$ po sinusima višestrukih uglova u intervalu $(0, \frac{\pi}{2})$.

18. Razviti funkciju $f(x) = \dfrac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0, \pi)$.

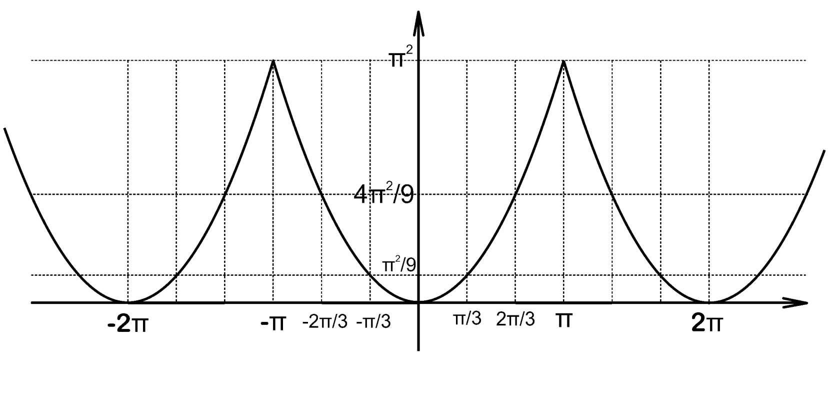


19. Dio grafika f-je $y = f(x)$ je prikazan na slici lijevo. Datu funkciju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{(-1)^n}{1 - 4n^2}$.

20. Funkciju definisanu grafikom pretvoriti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$.



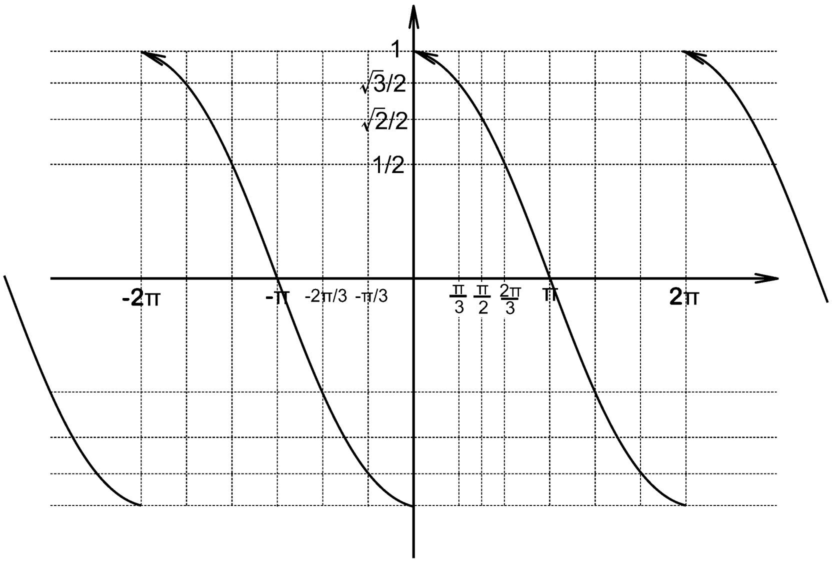
21. Razviti funkciju $f(x) = \dfrac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0, \pi)$. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{1}{n^2}$.



22. Funkciju definisanu grafikom razviti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje redova

(*a*) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...;$

(*b*) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$



23. Funkciju definisanu grafikom razviti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda
$$\frac{1}{1\cdot 3}-\frac{3}{5\cdot 7}+...+ \\ +\frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)}+....$$

4 Granične vrijednosti funkcija dviju promjenjivih

24. Neka je data funkcija $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ definisana na sljedeći način

$$f(x,y)=\begin{cases}\frac{(xy)^2}{(xy)^2+(x-y)^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$$

Odrediti da li sljedeći limesi postoje i izračunati one limese koji postoje:

(a) $\lim\limits_{x\to 0}[\lim\limits_{y\to 0} f(x,y)]$; $\lim\limits_{y\to 0}[\lim\limits_{x\to 0} f(x,y)]$;

(b) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

5 Neprekidnost funkcija dvije promjenjive

25. Ispitati neprekidnost funkcije $f(x,y)=\begin{cases}\dfrac{x^2y^3}{2x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

26. Ispitati neprekidnost funkcije $f(x,y)=\begin{cases}\frac{(x-1)^2 lnx}{(x-1)^2+y^2}, & (x,y)\neq(1,0)\\ 0, & (x,y)=(1,0)\end{cases}$.

6 Diferencijalni račun funkcija više realnih promjenjivih

27. Ako je $u=\dfrac{\varphi(x-y)+\psi(x+y)}{x}$ gdje su φ i ψ diferencijalne funkcije, izračunati

$$\frac{\partial}{\partial x}(x^2\frac{\partial u}{\partial x})-x^2\frac{\partial^2 u}{\partial y^2}.$$

28. Ako je $z=\dfrac{y}{f(x^2-y^2)}$ gdje je f diferencijalna funkcija, izračunati $\dfrac{1}{x}\cdot\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\cdot\dfrac{\partial z}{\partial y}$.

29. Ako je $z = e^y \varphi(ye^{\frac{x^2}{2y^2}})$ gdje je φ diferencijabilna funkcija, dokazati da je

$$(x^2 - y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xyz.$$

30. Provjeriti da li funkcija $z = arc\,tg\frac{x}{y}$, u kojoj je $x = u+v$, $y = u-v$, zadovoljava jednakost

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u-v}{v^2+u^2}.$$

31. Ako je $f(x) = \arcsin\frac{x}{y}$ gdje je $y = \sqrt{x^2+1}$ provjeriti da li je $\dfrac{\mathrm{d}f}{\mathrm{d}x} = \dfrac{1}{x^2+1}$.

32. Ako je $z = \ln(e^x + e^t)$ gdje je $x = t^3$ izračunati $\dfrac{\partial z}{\partial t}$ i $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

33. Provjeriti da li funkcija $u = \sin x + F(\sin y - \sin x)$, u kojoj je F diferencijabilna funkcija, zadovoljava jednakost $\dfrac{\partial u}{\partial y}\cos x + \dfrac{\partial u}{\partial x}\cos y = \cos x \cos y$.

34. Provjeriti da li funkcija $z = \varphi(x^2+y^2)$, u kojoj je φ diferencijabilna funkcija, zadovoljava jednakost

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$

35. Ako je $p = u^2 \ln v$ pri čemu je $u = \dfrac{x}{y}$ i $v = 3x - 2y$, odrediti $\dfrac{\partial p}{\partial x}$ i provjeriti da li vrijedi $\dfrac{\partial p}{\partial y} = -\dfrac{2xu}{vy^2}(v\ln v + y)$.

7 Tejlorova formula za funkcije dvije i veše promjenjivih

36. Razložiti funkciju $f(x,y) = arc\,tg(x^2y - 2e^{x-1})$ po formuli Tejlora u okolini tačke $M(1,3)$ do stepena drugog reda zaključno.

37. Funkciju $f(x,y) = arc\,tg\dfrac{x-y}{1+xy}$ razviti u Tejlorov red do članova četvrtog reda u okolini tačke $(0,0)$. Prikazati izgled opšteg člana.

8 Jednačina tangentne ravni i jednačina normale na površ

38. Odrediti jednačinu tangentne ravni na površ $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, koja je normalna na pravu $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$.

39. Naći jednačinu tangentne ravni elipsoida $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ koja na koordinatnim osama odsjeca jednake pozitivne odsječke.

40. Dokazati da tangentne ravni povrsi $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ $(a > 0)$ odsjecaju od koordinatnih osa odsjecke ciji je zbir jednak a.

41. Napisati jednačinu tangentne ravni i normale na površ $2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$ u tački $M(2,2,1)$.

9 Izvod funkcije u datom smijeru i gradijent funkcije

42. Izračunati izvod funkcije $u = x^2y^2 + z^2 - 3xyz$ u tački $T(1, 1, 2)$ u smjeru koji čini s koordinatnim osama uglove $\frac{\pi}{3}$, $\frac{\pi}{4}$ i $\frac{\pi}{6}$.

10 Ekstremi funkcija dvije i više promjenjih

43. Odrediti ekstreme funkcije $f(x, y) = x^2 - xy + y^2 - 2x - 2y$.

44. Naći ekstreme funkcije $z = x + y + 4 + 4\, sin\, x\, sin\, y$.

45. Naći ekstreme funkcije $z = (2x^2 + 3y^2)e^{-(x^2+y^2)}$.

46. Odrediti ekstreme funkcije $f(x, y) = xe^{y+x \sin y}$.

47. Naći ekstreme funkcije $z = x^3 + 4x^2y + xy^2 - 12xy - 3y^2$.

11 Dvostruki integrali

48. Izmjeniti poredak integracije u integralu $\int\limits_0^1 \mathrm{d}y \int\limits_y^{3y} f(x, y)\, \mathrm{d}x$.

49. Izmjeniti poredak integracije u integralu $\int\limits_0^1 \mathrm{d}x \int\limits_{x^3}^{x^2} f(x, y)\mathrm{d}y$.

50. Dati dvostruki integral $\int\limits_0^R \mathrm{d}x \int\limits_0^{\sqrt{R^2-x^2}} f(x, y)\, \mathrm{d}y$ iz pravougaonih koordinata transformisati na polarne koordinate.

51. Izračunati dvostruki integral $I = \iint\limits_D xy\, \mathrm{d}x\, \mathrm{d}y$, gdje je D oblast ograničena linijama $xy = 1$, $x + y = \frac{5}{2}$.

52. Izračunati $\iint_D \mathrm{d}x\, \mathrm{d}y$, ako je $D:\ y^2 - x^2 = 1,\ x^2 + y^2 = 4$.

53. Izračunati $I = \iint\limits_D (x^2 + y^2)\mathrm{d}x\mathrm{d}y$ gdje je D paralelogram sa stranicama $y = x$, $y = x + a$, $y = a$, $y = 3a$ $(a > 0)$.

12 Smjena promjenjivih u dvostrukom integralu

54. Izračunati dvostruki integral dat u polarnim koordinatama $I = \iint\limits_D \rho sin\varphi\, \mathrm{d}\rho\, \mathrm{d}\varphi$ gdje je oblast D

a) kružni sektor, ograničen linijama $\rho = a$, $\varphi = \frac{\pi}{2}$ i $\varphi = \pi$;

b) polukrug $\rho \leq 2acos\varphi$, $0 \leq \varphi \leq \frac{\pi}{2}$;

c) oblast između linija $\rho = 2 + cos\varphi$ i $\rho = 1$ (obavezno nacrtati izgled oblasti D u sve tri slučaja).

55. Izračunati integral $I = \iint\limits_D \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\mathrm{d}x\,\mathrm{d}y$ ako je D oblast data sa $x^2+y^2 \leq 1,\ y \geq 0$.

56. Izračunati dvostruki integral $I = \iint\limits_D \mathrm{d}x\,\mathrm{d}y$ ako je D oblast ogranicena lemniskatom $(x^2+y^2)^2 = a^2(x^2-y^2)$.

57. Izračunati dvostruki integral $\iint_D (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$ gdje je $D = \{(x,y) \in \mathbb{R} \mid x^2+y^2 \leq \frac{2}{3}(x+2y)\}$.

58. Dati dvostruki integral $\int\limits_{R/2}^{2R} dy \int\limits_{0}^{\sqrt{2Ry-y^2}} f(x,y)\,dx$ iz pravougaonih koordinata transformisati na polarne koordinate.

59. Izračunati dvostruki integral $\int\limits_{0}^{\frac{\sqrt{3}}{2}} dx \int\limits_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy$.

60. Izračunati dvostruki integral $\int\limits_{0}^{\sqrt{\frac{\pi}{2}}} \mathrm{d}x \int\limits_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}} \cos(x^2+y^2)\,\mathrm{d}y$.

61. Izračunati:

(a) dvostruki integral $\int_0^{2\pi} \mathrm{d}\varphi \int_0^a \rho^2 \sin^2\varphi\,\mathrm{d}\rho$;

(b) dvojni integral $\iint\limits_G \dfrac{xy\sqrt{1-x^2-y^2}}{2x^2+y^2}\mathrm{d}x\mathrm{d}y$ gdje je $G = \{(x,y) : x^2+y^2 \leq 1, x \geq 0, y \geq 0\}$.

62. Izračunati $I = \iint\limits_G \left(x + \dfrac{y^2}{x^2}\right)\mathrm{d}x\mathrm{d}y$ gdje je $G = \{(x,y) : x^2+y^2-2ax \leq 0, a > 0\}$.

63. Izračunati $\iint\limits_D y\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : 1 \leq x^2+y^2 \leq 2x,\ y \geq 0\}$.

64. Izračunati $\iint\limits_D x\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : 1 \leq x^2+y^2 \leq 2y,\ x \leq y,\ x \geq 0\}$.

65. Izračunati dvojni integral $I = \iint\limits_D \operatorname{arc\,tg}\dfrac{y}{x}\,\mathrm{d}x\mathrm{d}y$ gdje je

$D = \{(x,y) : 1 \leq x^2+y^2 \leq 9, \dfrac{x}{\sqrt{3}} \leq y \leq x\sqrt{3}\}$.

66. Izračunati $\iint\limits_D y\,\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : x^2+y^2 \leq 1,\ x^2+y^2 \leq 2x,\ y \geq 0\}$.

13 Trostruki integrali

67. Izračunati trojini integral $I = \iiint\limits_G \frac{1}{(1+z)^3}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, gdje je oblast G u prvom oktantu ograničena ravnima $x+y=1$, $z=x+y$, $x=0$, $y=0$, $z=0$.

68. Izračunati trostruki integral $I = \iiint\limits_D \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ ako je Ω oblast omeđena koordinatnim ravnima i sa ravni $x+y+z=1$.

69. Izračunati trostruki integral $I = \iiint\limits_\Omega z\,dx\,dy\,dz$ ako je Ω oblast ograničena površinama $y=x$, $y=2x$, $2x=1$, $x^2+y^2+z^2=1$, $z \geq 0$.

14 Računanje trostrukih integrala uvođenjem cilindričnih i sfernih koordinata

70. Uvođenjem cilindričnih koordinata izračunati trostruki integral $J = \iiint\limits_W (x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ gdje je oblast W ograničena površinom $3(x^2+y^2)+z^2=3a^2$.

71. Izračunati trostruki integral $K = \iiint\limits_T y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ gdje je oblast T ograničena površinama $y=\sqrt{x^2+z^2}$ i $y=h$, $h>0$.

72. Dati trojni integral $\iiint\limits_\Omega f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ transformisati na trostruki u cilindričnim koordinatama (sa određenim posebnim granicama integracije) ako je Ω oblast u prvom oktantu ograničen cilindrom $x^2+y^2=R^2$ i ravnima $z=0$, $z=1$, $y=x$ i $y=x\sqrt{3}$.

73. Dat je trostruki integral $\int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^2 r^3\,\mathrm{d}r \int\limits_0^{\sqrt{4-r^2}} \mathrm{d}z$ u cilindričnim koordinatama. Skicirati oblast integracije i izračunati taj integral prelazeći na sferne koordinate.

74. Izračunati trostruki integral $K = \iiint\limits_T ydxdydz$ gdje je oblast T ograničena površinama $y=\sqrt{x^2+z^2}$ i $y=h$, $h>0$.

75. Izračunati integral

$$\iiint\limits_\Omega \sqrt{x^2+y^2+z^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

gdje je $\Omega = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 \leq z,\ x^2+y^2 \leq z^2\}$.

76. Uvođenjem sfernih koordinata izračunati integral $\int\limits_0^1 \mathrm{d}x \int\limits_0^{\sqrt{1-x^2}} \mathrm{d}y \int\limits_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2\mathrm{d}z$.

77. Izračunati integral $\iiint\limits_V xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z$ gdje je oblast V ograničena sferom $x^2+y^2+z^2=1$ i ravnima $x=0$, $y=0$, $z=0$ u I oktantu.

15 Primjena dvostrukog i trostrukog integrala

78. Izračunati zapreminu tijela, koje je ograničeno sa površinama $z=y^2-x^2$, $z=0$, $y=\pm 2$.

79. Izračunati zapreminu tijela, ograničeno površinama $y=x^2$, $y=1$, $x+y+z=4$, $z=0$.

80. Izračunati zapreminu tijela ograničenog dijelom površi $(x^2+y^2+z^2)^3=\dfrac{a^6z^2}{x^2+y^2}$, $a>0$ u I oktantu.

81. Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

82. Izračunati zapreminu tijela ogranicenog valjkom $x^2+y^2=6x$ i ravnima $x-z=0$, $5x-z=0$.

83. Izračunati zapreminu tijela ograničenog ravninom $x0y$, valjkom $x^2+y^2=2ax$ i čunjem $x^2+y^2=z^2$.

84. Izračunati zapreminu tijela koju ravan $z=x+y$ odsijeca od paraboloida $z=x^2+y^2$.

85. Izračunati zapreminu dijela kugle $x^2+y^2+z^2=R^2$ koji se nalazi između dvije paralelne ravni $z=0$ i $z=a$ $(0<a<R)$.

86. Naći težište homogenog tijela ograničenog sa ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasiječen paralelopiped).

87. Izračunati zapreminu tijela ograničenog loptom $x^2+y^2+z^2=a^2$, cilindrom $x^2+y^2=ax$ i ravni $0xy$ koji se nalazi u gornjem poluprostoru.

16 Krivoliniski integral prve vrste (po luku)

88. Izračunati krivoliniski integral $I=\int\limits_L (4\sqrt[3]{x}-3\sqrt{y})dl$ između tački $E(-1;0)$ i $F(0;1)$

a) po pravoj EF;
b) po liniji astroide $x=cos^3t$, $y=sin^3t$.

89. Izračunati krivoliniski integral prve vrste

$$I=\oint\limits_C \sqrt{x^2+y^2}\,\mathrm{d}s$$

gdje je C krug $x^2+y^2=ax$, ($a>0$).

90. Izračunati krivoliniski integral $\int\limits_L (x-y)\,\mathrm{d}s$ po kružnoj liniji $x^2+y^2=ax$.

91. Izračunati krivoliniski integral prve vrste $\oint\limits_c (x+y)\,dS$ ako je $c:\begin{cases} x=a\,cos\varphi\,\sqrt{cos2\varphi}\\ y=a\,sin\varphi\,\sqrt{cos2\varphi}\\ -\frac{\pi}{4}\le\varphi\le\frac{\pi}{4}\end{cases}$ (kriva c je desna latica lemniskate $\rho=a\sqrt{cos2\varphi}$).

92. Izračunati krivoliniski integral $I = \int\limits_{AB} \frac{dl}{\sqrt{x^2+y^2}}$ po odsječku prave $x-2y=4$ od tačke $A(0;-2)$ do tačke $B(4;0)$.

93. Neka je A tačka u kojoj prava $2x-\sqrt{5}y-1=0$ siječe y-osu, a B tačka u kojoj data prava siječe x-osu. Izračunati krivolinijski integral prve vrste $\int\limits_{C} \frac{\mathrm{d}s}{\sqrt{x^2+y^2+1}}$, ako je C odsiječak date prave između tačaka A i B.

17 Krivoliniski integral druge vrste (po koordinatama)

94. Izračunati krivoliniske integrale

a) $\oint\limits_{-l} 2x\,dx-(x+2y)\,dy$ i b) $\oint\limits_{+l} ycosx\,dx+sinx\,dy$

po krivoj l, gdje je l trougao čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

95. Date su tačke $A(3;-6;0)$ i $B(-2;4;5)$. Izračunati krivoliniski integral $I = \int\limits_{c} xy^2\,dx + yz^2\,dy - zx^2\,dz$ gdje je c:

(a) duž koja spaja tačke O i B (O je koordinatni početak)

(b) kriva od A do B kruga zadan jednačinama $x^2+y^2+z^2=45$, $2x+y=0$.

96. Izračunati krivoliniski integral $I = \int\limits_{c}(x^2+y^2)\,dx + x^2y\,dy$ gdje je c kontura trapeza koga obrazuju prave $x=0$, $y=0$, $x+y=1$ i $x+y=2$.

97. Izračunati krivoliniski integral

$$I = \oint\limits_{C} z\,dz$$

duž krive koja nastaje kao presjek cilindra $\frac{(x-\frac{a}{2})^2}{\frac{a^2}{2}} + \frac{(y-\frac{b}{2})^2}{\frac{b^2}{2}} = 1$ i paraboloida $z = \frac{x^2}{a^2}+\frac{y^2}{b^2}$ orjentisana u pozitivnom smijeru ($a \geq b > 0$).

98. Izračunati krivoliniski integral $I = \oint\limits_{c} y\,dx + x^2\,dy$ duž krive koja nastaje kao presjek ravni $z=0$ i cilindra $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{x}{a}+\frac{y}{b}$ orjentisana u pozitivnom smijeru ($a \geq b > 0$).

99. Izračunati integral $I = \oint\limits_{c} y^2\,dx$ po krivoj koja nastaje kao presjek kugle $x^2+y^2+z^2=R^2$ i valjka $x^2+y^2=Rx$. (Mala pomoć: Da bi ste izračunali ovaj integral treba parametrizirati krivu c. Jedan od načina kako to možete postići je da krenete od parametrizacije kruga...)

100. Izračunati krivoliniske integrale (a) $I = \int\limits_{-l} 2xdx-(x+2y)dy$; (b) $I = \int\limits_{+l} ycosxdx+sinxdy$; gdje je l kontura trougla čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

101. Izračunati krivolinijski integral druge vrste

$$I = \oint\limits_{C} x\mathrm{d}y + x\mathrm{d}z$$

gdje je C kriva koja nastaje presjekom cilindrične površi $x^2 + y^2 = 2x$ i ravni $z = x$ pozitivno orjentisana ako se posmatra iz tačke $(0; 0; 1)$.

102. Izračunati krivoliniski integral druge vrste $I = \oint\limits_C (y - z)\mathrm{d}x + (z - x)\mathrm{d}y + (x - y)\mathrm{d}z$ gdje je C krug $x^2 + y^2 + z^2 = a^2$ $(a > 0)$, $y = x\,\mathrm{tg}\,\alpha$, $(0 < \alpha < \frac{\pi}{2})$ uzet u smjeru suprotnom kretanju kazaljnke na satu ako se posmatra sa pozitivnog dijela x-ose.

103. Izračunati vrijednost krivoliniskog integrala $I = \oint\limits_C y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z$ duž zatovorene krive C koja je dobijena kao presjek sljedećih površina: $x^2 + y^2 = r^2$ i $x^2 = rz$ $(r > 0)$. (Kriva C je orjentisana pozitivno ako se posmatra sa z-ose za $z > r$).

18 Green-Gausova formula

104. Pomoću Greenove formule izračunati krivoliniski integral

$$\oint\limits_c (x^2 y + \frac{1}{3}y^3 + ye^{xy})\,dx + (x + xe^{xy})\,dy$$

ako je c pozitivno rjentisana kontura određena linijama $y = \sqrt{1 - x^2}$, $y = 0$.

105. Izračunati krivoliniski integral $I = \int\limits_c (xy + x + y)dx + (xy + x - y)dy$ ako je $c : x^2 + y^2 = 3x$.

106. Pomoću Greenove formule izračunati integral $I = \int_c (xy + x + y)\,dx + (xy + x - y)\,dy$, ako je c kontura kruga $x^2 + y^2 = ax$ prijeđena u pozitivnom smislu.

107. Izračunati

$$I = \int\limits_C (e^{x+y}\sin 2y + x + y)\mathrm{d}x + (e^{x+y}(2\cos 2y + \sin 2y) + 2x)\mathrm{d}y$$

gdje je C kriva $y = \sqrt{2x - x^2}$, integracija se vrši od tačke $A(2; 0)$ do tačke $O(0; 0)$.

108. Izračunati

$$I = \int\limits_{\overset{\frown}{\mathrm{AO}}} (e^x \sin y - my)\mathrm{d}x + (e^x \cos y - m)\mathrm{d}y$$

gdje je $\overset{\frown}{\mathrm{AO}}$ gornji polukrug $x^2 + y^2 = ax$, $y \geq 0$ $(a > 0)$ orjentisan od tačke $A(a; 0)$ do tačke $O(0; 0)$.

19 Primjena krivoliniskog integrala druge vrste: Računanje površine ravne figure

109. Uz pomoć krivoliniskog integrala druge vrste, izračunati površinu, ograničenu kardioidom $x = 2\,cost - cos2t$, $y = 2sint - sin2t$.

110. Izračunati pomoću krivoliniskog integrala druge vrste površinu ravne figure ograničene konturom

$$\mathrm{c} : \begin{cases} x = a(t - sin\,t) \\ y = a(1 - cos\,t) \\ \quad 0 \leq t \leq 2\pi \end{cases}$$

20 Nezavisnost krivoliniskog integrala od vrste konture. Određivanje primitivnih funkcija

111. Izračunati krivoliniski integral $\int\limits_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$ duž puta koji ne prolazi kroz koordinatni početak.

112. Izračunati krivoliniski integral $\int\limits_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2}$ duž puta koji ne siječe osu $0y$.

21 Površinski integral prve vrste

113. Izračunati površinski integral $I = \iint\limits_S xyz\,dS$, ako je S dio ravni $x + y + z = 1$ u I oktantu.

114. Izračunati površinski integral $\int\int_S 3z\,dS$ gdje je S površina paraboloida $z = 2-(x^2+y^2)$ iznad xy-ravni.

115. Izračunati površinski integral

$$\iint\limits_{(S)} \sqrt{-x^2+4}\,dS,$$

gdje je (S) omotač površi $\frac{x^2}{4} + \frac{y^2}{4} = \frac{z^2}{9}$, $0 \le z \le 3$.

116. Izračunati površinski integral prvog tipa $I = \iint\limits_W (x^2+y^2)\mathrm{d}s$ gdje je W-površina dijela paraboloida $x^2+y^2 = 2z$ koju odsjeca ravan $z = 1$ (dio paraboloida ispod date ravni).

117. Izračunati površinski integral $I = \iint\limits_S \frac{\mathrm{d}S}{(1+z)^2}$ ako je S sfera $x^2+y^2+z^2 = 1$.

22 Površinski integral druge vrste

118. Izračunati površinski integral drugog tipa (po koordinatama) $I = \iint\limits_\sigma \sqrt[4]{x^2+y^2}dxdy$ gdje je σ donja strana kruga $x^2+y^2 \le a^2$.

119. Izračunati površinski integral

$$K = \oiint\limits_{-W} y\,dx\,dz$$

gdje je W-površina tetraedra ograničenog ravnima $x+y+z = 1$, $x = 0$, $y = 0$ i $z = 0$.

120. Izračunati površinski integral $\iint\limits_T 2\,dxdy + y\,dxdz - x^2 z\,dydz$ gdje je T vanjska strana elipsoida $4x^2 + y^2 + 4z^2 = 4$ koji se nalazi u prvom oktantu.

121. Izračunati površinski integral $\iint\limits_S xy^3 z\,dx\,dy$ ako je S vanjska strana sfere $x^2+y^2+z^2 = 4$ u prvom oktantu.

122. Izračunati površinski integral druge vrste

$$I = \iint\limits_S xyz\,\mathrm{d}x\mathrm{d}y$$

gdje je S spoljna strana dijela sfere $x^2 + y^2 + z^2 = 1$, $x \geq 0$, $y \geq 0$.

123. Izračunati površinski integral drugog tipa (po koordinatama) $I = \iint\limits_\sigma \sqrt[4]{x^2 + y^2}\mathrm{d}x\mathrm{d}y$ gdje je σ donja strana kruga $x^2 + y^2 \leq a^2$.

124. Izračunati

$$I = \iint\limits_{S+} \left(\frac{1}{x}\mathrm{d}y\mathrm{d}z + \frac{1}{y}\mathrm{d}z\mathrm{d}x + \frac{1}{z}\mathrm{d}x\mathrm{d}y\right)$$

gdje je $S+$ spoljašnja strana jedinične sfere (zadatak uraditi bez upotrebe teoreme Gauss-Ostrogradskog - zadatak se i ne može uraditi uz pomoć navedene teoreme zato što ne ispunjavaju sve uslove teoreme).

125. Izračunati površinski integral $I = \iint\limits_{S+} y^2\mathrm{d}y\mathrm{d}z + (y^2 + x^2)\mathrm{d}z\mathrm{d}x + (y^2 + x^2 + z^2)\mathrm{d}x\mathrm{d}y$ gdje je $S+$ spoljašnja strana polusfere $x^2 + y^2 + z^2 = 2Rx$, $z > 0$ (za fiksirano $R > 0$).

126. Data je kriva c koja je dobijena kao presjek površina $x^2 + y^2 = r^2$ i $x^2 = rz$ $(r > 0)$. Izračunati površinski integral $\iint\limits_S \mathrm{d}x\mathrm{d}y$ gdje je S gornja strana površine koju zatvara kriva c.

23 Primjena površinskog integrala

127. Izračunati $\iint_S dS$, ako je S površina djela sfere $S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = a^2\}$ koja se nalazi u unutrašnjosti cilindra $S_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, z \in \mathbb{R}\}$, $b < a$.

128. Neka je S površina tijela koje je dobijeno presjekom dva cilindra $S_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + z^2 = a^2, y \in \mathbb{R}\}$ i $S_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, y^2 + z^2 = a^2, x \in \mathbb{R}\}$.
Izračunati $\iint\limits_S dS$.

129. Izračunati površinu dijela površi $S : z^2 = 2xy$ određene u prvom oktantu u presjeku sa ravnima: $x = 0$, $y = 0$ i $x + y = 1$.
Uputa: $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}\,dx$, $B(\frac{3}{2}, \frac{3}{2}) = \frac{\pi}{8}$, $B(\frac{1}{2}, \frac{5}{2}) = \frac{3\pi}{8}$.

130. Izračunati površinu dijela lopte $x^2+y^2+z^2 = 3a^2$ koja se nalazi ispod parabole $x^2+y^2 = 2az$ a iznad $x0y$ ravni.

131. Izračunati površinu onog dijela kupe $z^2 = x^2+y^2$ koji se nalazi unutar valjka $x^2+y^2 = 2x$.

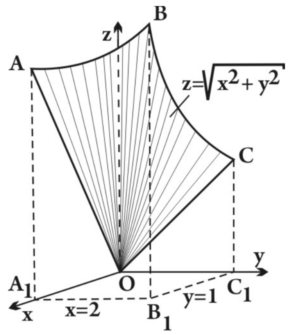
132. Odrediti površinu koju cilindar $x^2 + y^2 = ax$ isjeca na lopti $x^2 + y^2 + z^2 = a^2$ iznad ravni Oxy.

24 Formula Stoksa

133. Uz pomoć formule Stoksa, izračunati krivoliniski integral

$$\oint_l e^x dx + z(x^2+y^2)^{\frac{3}{2}} dy + yz^3 dz$$

gdje je l-zakrivljena linija OCBAO (vidi sliku) dobijena presjekom površina $z = \sqrt{x^2+y^2}$, $x = 0$, $x = 2$, $y = 0$, $y = 1$.



134. Uz pomoć formule Stoksa, izračunati krivolinijski integral $I = \oint_L x^2y^3 \mathrm{d}x + \mathrm{d}y + z\mathrm{d}z$ gdje je L krug dat sa $x^2 + y^2 = r^2$ i $z = 0$ (r>0). (L je pozitivno orjentisana kriva ukoliko se posmatra sa pozitivnog dijela z-ose.)

25 Formula Gaus-Ostrogradskog

135. Uz pomoć formule Gauss-Ostrogradski izračunati površinski integral

$$\oiint_S 4x^3 \mathrm{d}y\mathrm{d}z + 4y^3 \mathrm{d}x\mathrm{d}z - 6z^4 \mathrm{d}x\mathrm{d}y$$

gdje je S vanjska strana cilindra $x^2 + y^2 = a^2$ koji se nalazi između ravni $z = 0$ i $z = h$.

136. Pomoću formule Gauss-Ostrogradski izračunati površinski integral

$$I = \oiint_S xz\,\mathrm{d}y\mathrm{d}z + xy\,\mathrm{d}z\mathrm{d}x + yz\,\mathrm{d}x\mathrm{d}y,$$

ako je S vanjska strana tijela koje pripada prvom oktantu i ograničeno je cilindrom $x^2+y^2 = 1$, te ravnima $x = 0$, $y = 0$, $z = 0$, $z = 2$.

137. Izračunati površinski integral $I = \iint_{S+} x^2 \mathrm{d}y\mathrm{d}z + y^2 \mathrm{d}z\mathrm{d}x + z^2 \mathrm{d}x\mathrm{d}y$ gdje je $S+$ spoljašnja strana kupe određena omotačem $z^2 = x^2 + y^2$, $0 \le z \le h$ i osnovom $x^2 + y^2 \le h^2$, $z = h$ za fiksirano $h > 0$.

26 Integrali ovisni o parametru

138. Prvo izračunati integral $I = \int\limits_0^\infty e^{-x}\sin(\alpha x)\mathrm{d}x$ pa poslije toga dobijeni rezultat iskoristiti i koristeći metodu diferenciranja po parametru izračunati

$$G(\alpha) = \int\limits_0^\infty xe^{-x}\cos(\alpha x)\mathrm{d}x$$

139. Date su vrijednosti dva integrala ($\alpha > 0$)

$$\int_0^\infty \frac{\cos\alpha x}{1+x^2}\mathrm{d}x = \frac{\pi}{2}e^{-\alpha}, \qquad \int_0^\infty \frac{\sin\alpha x}{x}\mathrm{d}x = \frac{\pi}{2}.$$

Koristeći date jednakosti, uz pomoć metode diferenciranja po parametru izračunati $\int_0^\infty \frac{\sin\alpha x}{x(1+x^2)}\mathrm{d}x$.

140. Metodom diferenciranja po parametru izračunati integral $\int\limits_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}}\mathrm{d}x \quad (a^2 < 1)$ (mala pomoć: možda ćete naći korisno da u rješavanju integrala iskoristie smjene $x = \sin t$ ili $\operatorname{tg} t = z$).

141. Metodom diferenciranja po parametru izračunati integral $\int\limits_0^1 \frac{\operatorname{arc\,tg} ax}{x\sqrt{1-x^2}}\mathrm{d}x$ (mala pomoć: možda ćete naći korisno da u rješavanju integrala iskoristie smjene $x = \sin t$ ili $\operatorname{tg} t = z$).

27 Vektorska teorija polja

142. Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v} = 2x(y^2+z^2)\vec{i} + 2y(x^2+z^2)\vec{j} + 2z(x^2+y^2)\vec{k}.$$

143. Odrediti brojeve a i b tako da vektorsko polje $\vec{v} = (yz + axy, xz + bx^2 + yz^2, axy + y^2z)$ bude potencijalno i za dobijeno polje izračunati njegovu cirkulaciju duž pravoliniske konture od tačke $A(1;1;1)$ prema tački $B(2;2;2)$

144. Neka funkcije $g, h : \mathbb{R}^3 \longrightarrow \mathbb{R}$ ispinjavaju

$$\Delta g(x,y,z) = 0 \quad \text{i} \quad \Delta h(x,y,z) = 0$$

gdje je $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ Laplace-ov operator. Za funkciju $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ datu sa

$$f(x,y,z) = g(x,y,z) + (x^2+y^2+z^2)h(x,y,z)$$

izračunati $\Delta\Delta f(x,y,z)$.

145. Pokazati da je vektorsko polje $\vec{v} = (2x+y+z, x+2y+z, x+y+2z)$ potencijalno i naći njegov potencijal.

146. Dokazati da je vektorsko polje $\vec{v} = (z\cos zx - y\sin x, \cos x, x\cos zx)$ potencijalno i izračunati cirkulaciju tog polja duž prave od tačke $O(0,0,0)$ do tačke $A(1,2,\pi)$.

28 Cirkulacija i fluks vektorskog polja

147. Izračunati cirkulaciju vektorskog polja $\vec{v} = (1, xy^2, yz^2)$ duž konture $x^2 + 2y^2 = 4$, $z = 2x$.

148. Izračunati cirkulaciju polja $\vec{v} = x\vec{i} + y\vec{j} + (x+y-1)\vec{k}$ duž odsječka prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.

149. Data su skalarna polja $f = xyz$, $g = xy + yz + zx$.

(a) Formirati vektorska polja $\vec{a} = \text{grad} f$, $\vec{b} = \text{grad} g$ i ispitati prirodu vektorskog polja $\vec{a} \times \vec{b}$ (drugim riječima odgovoriti na pitanje da li je polje $\vec{a} \times \vec{b}$ potencijalno ili solenoidno).

(b) Izračunati $\int\limits_C (\vec{a} \times \vec{b}) \mathrm{d}r$, gdje je C duž koja spaja tačke $O(0,0,0)$ i $B(1,2,3)$.

150. Izračunati fluks vektorskog polja

$$\vec{v} = (x, -y^2, x^2 + z^2 - 1)$$

po unutrašnjoj strani sfere $x^2 + y^2 + z^2 = 1$.

(#) Izračunati integrale

a) $\int_2^3 3x^2 dx$; b) $\int_0^4 (1+e^{\frac{x}{4}}) dx$; c) $\int_{-1}^7 \frac{dt}{\sqrt{3t+4}}$;

d) $\int_0^{\pi/2a} (x+3) \sin ax$

Rj. a)

$$\int_2^3 3x^2 dx = 3\int_2^2 x^2 dx = 3\cdot \frac{x^3}{3}\Big|_2^3 = x^3\Big|_2^3 = 3^3 - 2^3 = 27-8 = 19$$

b)

$$\int_0^4 (1+e^{\frac{x}{4}}) dx = \int_0^4 dx + \int_0^4 e^{\frac{x}{4}} dx = \int_0^4 dx + 4\int_0^4 e^{\frac{x}{4}} d\left(\frac{x}{4}\right) =$$

$$= x\Big|_0^4 + 4\, e^{\frac{x}{4}}\Big|_0^4 = (4-0) + 4\left(e^{\frac{4}{4}} - e^{\frac{0}{4}}\right) = 4 + 4e - 4 = 4e$$

c)

$$\int_{-1}^7 \frac{dt}{\sqrt{3t+4}} = \int_{-1}^7 (3t+4)^{-\frac{1}{2}} dt = \left| \begin{array}{l} d(3t+4) = 3\,dt \\ dt = \frac{1}{3} d(3t+4) \end{array} \right| =$$

$$= \frac{1}{3}\int_{-1}^7 (3t+4)^{-\frac{1}{2}} d(3t+4) = \frac{2}{3}(3t+4)^{\frac{1}{2}}\Big|_{-1}^7 = \frac{2}{3}\left(\sqrt{25} - \sqrt{1}\right) = \frac{8}{3}$$

d)

$$\int_0^{\frac{\pi}{2a}} (x+3)\sin ax\, dx = \left| \begin{array}{ll} u = x+3 & dv = \sin ax\, dx \\ du = dx & v = \frac{1}{a}\int \sin ax\, d(ax) = \frac{-1}{a}\cos ax \end{array} \right| =$$

$$= -\frac{1}{a}(x+3)\cos ax\Big|_0^{\frac{\pi}{2a}} + \frac{1}{a}\int_0^{\pi/2a} \cos ax\, dx = -\frac{1}{a}\Big[\left(\frac{\pi}{2a}+3\right)\underbrace{\cos\frac{\pi}{2}}_{=0} - 3\underbrace{\cos 0}_{=1}\Big] +$$

$$+ \frac{1}{a}\cdot\frac{1}{a}\int_0^{\pi/2a} \cos ax\, d(ax) = \frac{3}{a} + \frac{1}{a^2}\underbrace{\sin ax\Big|_0^{\frac{\pi}{2a}}}_{\sin\frac{\pi}{2} - \sin 0} = \frac{3}{a} + \frac{1}{a^2} = \frac{1+3a}{a^2}$$

Zamena promenljivih u određenom integralu

$$\int_a^b f(x)\,dx = \left| \begin{array}{lll} x=\varphi(t) & x=a \Rightarrow a=\varphi(\alpha) \Rightarrow & t=\alpha \\ dx=\varphi'(t)\,dt & x=b \Rightarrow b=\varphi(\beta) \Rightarrow & t=\beta \end{array} \right|$$

$$= \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt = \int_\alpha^\beta F(t)\,dt$$

(#) Izračunati integrale

a) $\displaystyle\int_0^5 \frac{x\,dx}{\sqrt{1+3x}}$; b) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{dx}{e^x-e^{-x}}$; c) $\displaystyle\int_1^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}}$; d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

Rj.

a) $$\int_0^5 \frac{x\,dx}{\sqrt{1+3x}} = \left| \begin{array}{ll} 1+3x=t^2 & 3\,dx=2t\,dt \\ \sqrt{1+3x}=t & dx=\frac{2}{3}t\,dt \\ 3x=t^2-1 & \\ x=\frac{t^2-1}{3} & x\Big|_0^5 \Rightarrow t\Big|_1^4 \end{array} \right| = \int_1^4 \frac{\frac{t^2-1}{3}\cdot\frac{2}{3}t\,dt}{t} =$$

$$= \frac{2}{9}\int_1^4 (t^2-1)\,dt = \frac{2}{9}\left(\frac{t^3}{3}\Big|_1^4 - t\Big|_1^4\right) = \frac{2}{9}\left(\frac{1}{3}(64-1)-(4-1)\right) =$$

$$= \frac{2}{9}\left(\frac{63}{3}-3\right) = \frac{2}{9}(21-3) = 4$$

b) $$\int_{\ln 2}^{\ln 3} \frac{dx}{e^x-e^{-x}} = \left| \begin{array}{ll} e^x=t & e^{-x}=t^{-1}=\frac{1}{t} \\ x=\ln t & \\ dx=\frac{dt}{t} & x\Big|_{\ln 2}^{\ln 3} \Rightarrow t\Big|_2^3 \end{array} \right| = \int_2^3 \frac{\frac{dt}{t}}{t-\frac{1}{t}} = \int_2^3 \frac{\frac{dt}{t}}{\frac{t^2-1}{t}} =$$

$$= \int_2^3 \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\Bigg|_2^3 = \frac{1}{2}\left(\ln\frac{2}{4}-\ln\frac{1}{3}\right) = \frac{1}{2}\cdot\ln\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\ln\frac{3}{2}}{2}$$

c) $$\int_1^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}} = \left| \begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \quad x\Big|_1^{\sqrt{3}} \Rightarrow t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ x^3 = 8\sin^3 t \\ \sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = \sqrt{4(1-\sin^2 t)} \end{array} \right| =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(8\sin^3 t+1)\,2\cos t\,dt}{4\sin^2 t\sqrt{4\cos^2 t}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin^3 t+1}{4\sin^2 t}\,dt =$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin t\,dt + \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dt}{\sin^2 t} = -2\cos t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \frac{1}{4}\operatorname{ctg} t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$= -2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) - \frac{1}{4}\left(\frac{\sqrt{3}}{3} - \sqrt{3}\right) = \frac{7}{2\sqrt{3}} - 1$$

d) $$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \left| \begin{array}{ll} \operatorname{tg}\frac{x}{2} = z & dx = \frac{2dz}{1+z^2} \\ \cos x = \frac{1-z^2}{1+z^2} & x\Big|_0^{\frac{\pi}{2}} \Rightarrow z\Big|_0^1 \end{array} \right| =$$

$$= \int_0^1 \frac{\frac{2dz}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} = 2\int_0^1 \frac{dz}{z^2+3} = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{z}{\sqrt{3}}\Big|_0^1 =$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

(7) Dokazati da za parnu f-ju $f(x)$ vrijedi

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

dok za neparnu f-ju $f(x)$ vrijedi $\int_{-a}^{a} f(x)\,dx = 0$.

Rj. Prvo rastavimo interval $[-a,a]$ na dva dijela $[-a,0]$ i $[0,a]$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \ldots (*)$$

Posmatrajmo sad $\int_{-a}^{0} f(x)\,dx$. Ako uvedemo smjenu $x=-z$ imamo da je $dx=-dz$ i $z_1=a$ za $x_1=-a$, $z_2=0$ za $x_2=0$

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-z)\,dz = \int_{0}^{a} f(-z)\,dz = \left| \begin{matrix} \text{nove prom.} \\ x=z \end{matrix} \right| = \int_{0}^{a} f(-x)\,dx$$

Prema tome $(*)$ sad postaje

$$I = \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} [f(x)+f(-x)]\,dx$$

Za parnu f-ju znamo da $f(-x)=f(x)$ dok je za neparnu f-ju $f(-x)=-f(x)$. Prema tome

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx, & \text{ako je } f(x) \text{ parna f-ja} \\ 0, & \text{ako je } f(x) \text{ neparna f-ja} \end{cases}$$

(#) Izračunati dužinu luka polukubičnog paraboloida $y^2=(x-1)^3$ između tački $A(2;-1)$ i $B(5;-8)$.

Rj.

Dužinu luka krive $y=f(x)$ između tački $(a;f(a))$ i $(b;f(b))$ računamo pomoću formule

$$l=\int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

U našem slučaju

$$y^2=(x-1)^3$$

$$y=\pm\sqrt{(x-1)^3}=\pm(x-1)^{\frac{3}{2}} \Rightarrow y'=\pm\frac{3}{2}(x-1)^{\frac{1}{2}}$$

$$L_{AB}=\int_{x_A}^{x_B}\sqrt{1+(y')^2}\,dx=\int_2^5\sqrt{1+\frac{9}{4}(x-1)}\,dx=\frac{1}{2}\int_2^5\sqrt{9x-5}\,dx=$$

$$=\left|\begin{array}{l} d(9x-5)=9\,dx \\ dx=\frac{1}{9}\,d(9x-5)\end{array}\right|=\frac{1}{18}\int_2^5(9x-5)^{\frac{1}{2}}\,d(9x-5)=$$

$$=\frac{1}{27}(9x-5)^{\frac{3}{2}}\Big|_2^5=\ldots=\frac{80}{27}\sqrt{10}-\frac{13}{27}\sqrt{13}$$

traženo rešenje

$$\approx 7{,}6337$$

(#) Izračunati dužinu luka jednog svoda cikloide $x = a(t - \sin t)$, $y = a(1 - \cos t)$ (za jedan svod cikloide parametar t uzima vrijednosti od 0 do 2π).

Rj.

Dužina luka krive $\begin{cases} x = x(t) \\ y = y(t) \\ t_1 \le t \le t_2 \end{cases}$ se računa po formuli

$$l = \int_{t_1}^{t_2} \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt .$$

U našem slučaju

$$\dot{x} = \frac{dx}{dt} = a(1 - \cos t) \; ; \quad \dot{y} = \frac{dy}{dt} = a \sin t$$

$$\sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t}\, dt =$$

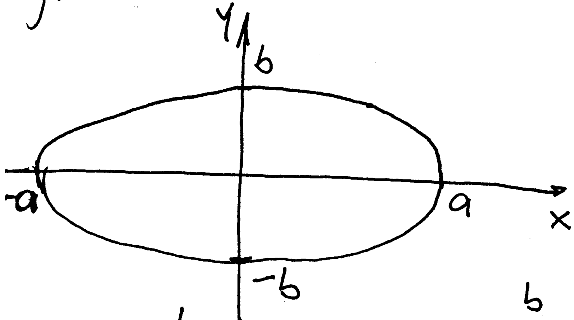
$$= a\sqrt{2(1-\cos t)}\, dt = a\sqrt{4\sin^2\frac{t}{2}}\, dt = 2a \sin\frac{t}{2}\, dt$$

$$L = 2a\int_0^{2\pi} \sin\frac{t}{2}\, dt = 4a\int_0^{2\pi} \sin\frac{t}{2}\, d\frac{t}{2} = -4a\cos\frac{t}{2}\Big|_0^{2\pi} = \ldots = 8a$$

traženo rješenje

\# Izračunati zapreminu tijela koje nastaje rotacijom krive $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ oko y-ose.

Rj.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 = a^2 \cdot \left(1 - \frac{y^2}{b^2}\right)$$

$$x^2 = a^2 \cdot \frac{b^2 - y^2}{b^2}$$

$$x^2 = \frac{a^2}{b^2} \cdot (b^2 - y^2)$$

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2}$$

$$V_y = \pi \int_{-b}^{b} [f(y)]^2 \, dy = \pi \int_{-b}^{b} \frac{a^2}{b^2}(b^2 - y^2)\, dy = 2\pi \frac{a^2}{b^2} \int_0^b (b^2 - y^2)\, dy =$$

↖ Parna f-ja (simetrična u odnosu na y-osu)

$$= 2\pi \frac{a^2}{b^2} \left(b^2 y \Big|_0^b - \frac{y^3}{3} \Big|_0^b \right) = 2\pi \frac{a^2}{b^2} \left(b^3 - \frac{1}{3} b^3 \right) = 2\pi \frac{a^2}{b^2} \cdot \frac{2}{3} b^3 = \frac{4\pi a^2 b}{3}.$$

Figura u ravni ograničena parabolom $y=4-x^2$ i pravama $y \geq 3x$, $y \geq 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

Rj.

$$y = 4 - x^2$$
$$y = 3x$$

$$3x = 4 - x^2$$

$$x^2 + 3x - 4 = 0$$

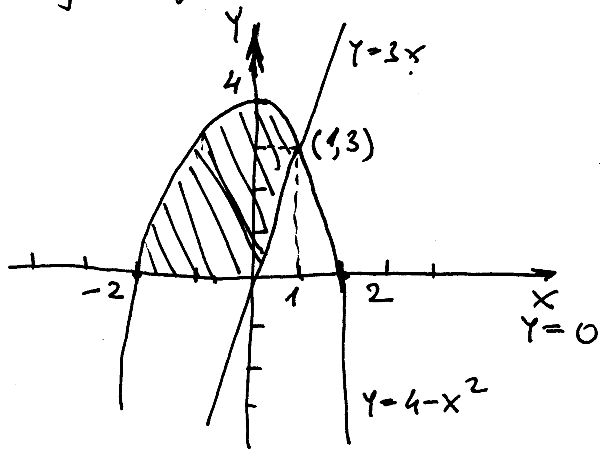
$$D = 9 + 16 = 25$$

$$x_{1,2} = \frac{-3 \pm 5}{2}$$

$x_1 = -4 \Rightarrow y_1 = -12$

$x_2 = 1 \Rightarrow y_2 = 3$

$A(-4,-12)$ i $B(1,3)$ su tačke presjeka prave i parabole



$$V_x = V_1 - V_2, \quad V_1 = \pi \int_{-2}^{1} (4-x^2)^2 dx, \quad V_2 = \pi \int_0^1 (3x)^2 dx$$

$$V_1 = \pi \int_{-2}^{1} (16 - 8x^2 + x^4)\, dx = \pi \left(16x \Big|_{-2}^{1} - 8 \frac{x^3}{3} \Big|_{-2}^{1} + \frac{x^5}{5} \Big|_{-2}^{1} \right) = \pi \left(16 \cdot 3 - \frac{8}{3} \cdot 9 + \frac{1}{5} \cdot 33 \right)$$

$1-(-2)$ $\quad 1^3-(-2)^3$ $\quad 1^5-(-2)^5$

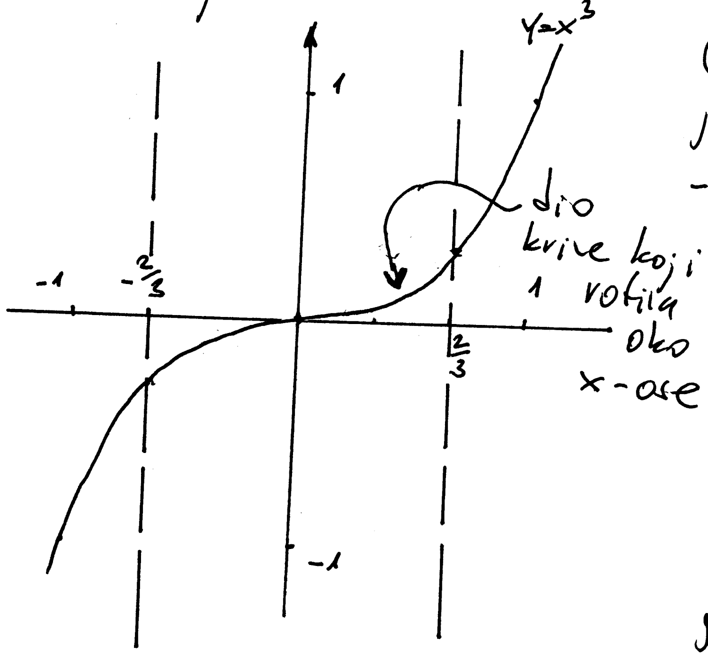
$$= \pi \left(48 - 24 + \frac{33}{5} \right) = \frac{153}{5} \pi$$

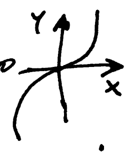
$$V_2 = \pi \cdot 9 \int_0^1 x^2 dx = 9\pi \frac{x^3}{3} \Big|_0^1 = 3\pi(1-0) = 3\pi$$

$$V = V_1 - V_2 = \frac{158}{5}\pi - 3\pi = \frac{158\pi - 15\pi}{5} = \frac{138}{5}\pi$$

Izračunati površinu omotača tijela koje nastaje kada dio krive $y=x^3$, koji se nalazi između pravih $x=-\frac{2}{3}$ i $x=\frac{2}{3}$, rotira oko x-ose.

Rj. Skicirajmo sliku



Znamo da kriva $y=x^3$ izgleda ovako 

U našem slučaju dovoljno ju je nacrtati u granicama od -1 do 1.

Prisjetimo se formule

$$P = 2\pi \int_{t_1}^{t_2} |\mu(t)| \sqrt{[\eta'(t)]^2 + [\mu'(t)]^2}\, dt$$

gdje je $c: \begin{cases} x=\eta(t) \\ y=\mu(t) \\ t_1 \le t \le t_2 \end{cases}$

Naša kriva je

$c: \begin{cases} y=x^3 \\ -\frac{2}{3} \le x \le \frac{2}{3} \end{cases}$ ili napisano u drugačijem obliku $c: \begin{cases} x=t \\ y=t^3 \\ -\frac{2}{3} \le t \le \frac{2}{3} \end{cases}$

Kako je $y' = 3x^2$ imamo

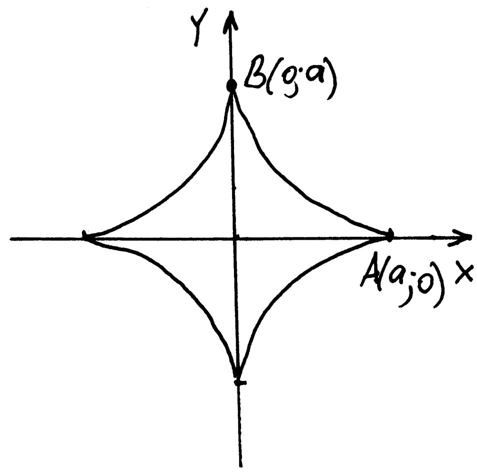
$$P = 2\pi \int_{-2/3}^{2/3} |x^3| \sqrt{1+3x^2}\, dx = 2\pi \cdot 2 \int_0^{2/3} \underbrace{x^3}_{x^2 \cdot x} \sqrt{1+(3x^2)^2}\, dx = 4\pi \int_0^{2/2} x^3 \sqrt{1+9x^4}\, dx$$

$$= \left| \begin{array}{l} 1+9x^4 = z \\ 36x^3 dx = dz \\ x^3 dx = \frac{1}{36} dz \end{array} \quad x\Big|_0^{\frac{2}{3}} \Rightarrow z\Big|_1^{\frac{25}{9}} \right| = 4\pi \cdot \frac{1}{36} \int_1^{25/9} t^{\frac{1}{2}} dt = \frac{\pi}{9} \cdot \frac{2}{3} t^{\frac{3}{2}} \Big|_1^{\frac{25}{9}} =$$

$$= \frac{2\pi}{27} \left(\frac{125}{27} - 1 \right)$$ traženo rješenje

Izračunati površinu omotača tijela koje nastaje kada astroida $x = a\cos^3 t$, $y = a\sin^3 t$ rotira oko x-ose.

Uputa: Grafik astroide izgleda ovako



Rj:

Tijelo koje će nastati prilikom rotacije astroide će biti simetrično u odnosu na x i y-osu.

Prema tome dovoljno je rotirati samo jedan luk astroide oko x-ose i dobijeni rezultat pomnožiti sa 2.

Prisjetimo se formule

$$P = 2\pi \int_{t_1}^{t_2} |\mu(t)| \sqrt{[\eta'(t)]^2 + [\mu'(t)]^2}\, dt \quad \text{gdje je:} \quad C: \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{cases}$$

U našem slučaju

$$C: \begin{cases} x = a\cos^3 t \\ y = a\sin^2 t \\ 0 \le t \le \frac{\pi}{2} \end{cases} \qquad \begin{aligned} x' &= 3a\cos^2 t(-\sin t) \\ y' &= 3a\sin^2 t\cos t \end{aligned}$$

$$x'^2 + y'^2 = 9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t = 9a^2\cos^2 t\sin^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{=1}$$

$$\sqrt{x'^2 + y'^2} = 3a\cos t\sin t$$

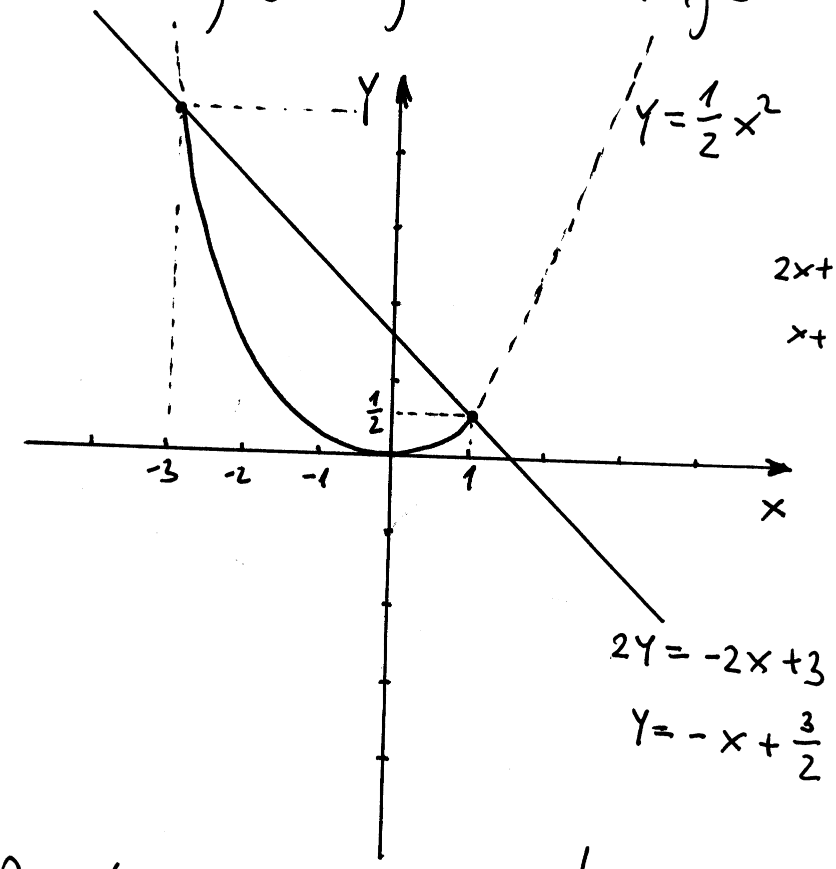
$$\frac{1}{2}P = 2\pi\int_0^{\pi/2} a\sin^3 t \cdot 3a\cos t\sin t\, dt = 6a^2\pi\int_0^{\pi/2}\sin^4 t\cos t\, dt =$$

$$= 6a^2\pi\int_0^{\pi/2}\sin^4 t\, d(\sin t) = 6a^2\pi\,\frac{1}{5}\sin^5 t\Big|_0^{\pi/2} = \frac{6a^2\pi}{5}$$

Prema tome $P = \dfrac{12a^2\pi}{5}$ tražena površina

(#) Figura u ravni ograničena linijama $2y = x^2$ i $2x + 2y - 3 = 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

Rj. Skicirajmo dvije date linije



$2x + 2y - 3 = 0 \; /:2$

$x + y - \frac{3}{2} = 0$

$y = \frac{1}{2}x^2$

$$\begin{aligned} 2y &= x^2 \\ 2x + 2y - 3 &= 0 \end{aligned}$$

$x^2 + 2x - 3 = 0$

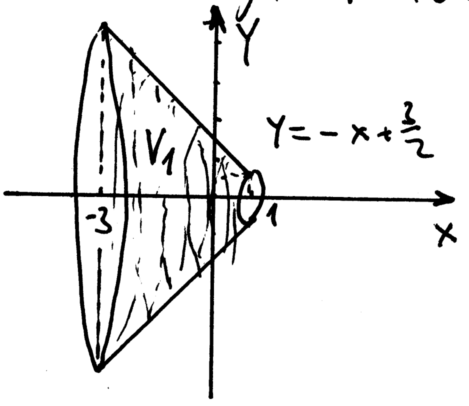
$(x+3)(x-1) = 0$

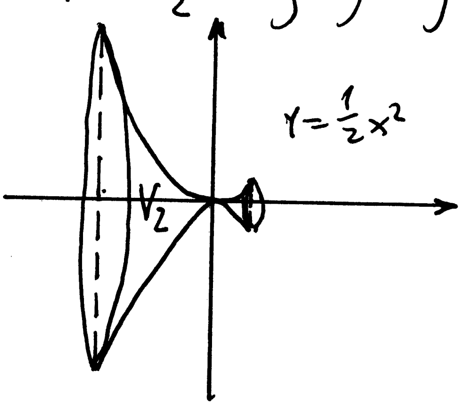
$x_1 = -3 \Rightarrow y_1 = \frac{9}{2}$

$x_2 = 1 \Rightarrow y_2 = \frac{1}{2}$

Prisjetimo se: $V_x = \pi \int_a^b y^2 \, dx$ je zapremina tijela kada f-ja $y = f(x)$ rotira oko x-ose, za $a \le x \le b$

U našem slučaju imademo $V = V_1 - V_2$ gdje je





$$V_1 = \pi \int_{-3}^{1} \left(-x + \frac{3}{2}\right)^2 dx = \pi \int_{-3}^{1} \left(x^2 - 3x + \frac{9}{4}\right) dx =$$

$$= \frac{1}{3} x^3 \Big|_{-3}^{1} - \frac{3}{2} x^2 \Big|_{-3}^{1} + \frac{9}{4} x \Big|_{-3}^{1} = \ldots = \frac{91}{3} \pi$$

integral u V_1 smo mogli izračunati i na drugi način

$$V_1 = \pi \int_{-3}^{1} \left(-x + \frac{3}{2}\right)^2 dx = \pi \int_{-3}^{1} (-1)^2 \left(x - \frac{3}{2}\right)^2 d\left(x - \frac{3}{2}\right) =$$

$$= \pi \cdot \frac{1}{3} \left(x - \frac{3}{2}\right)^3 \Big|_{-3}^{1} = \ldots = \frac{91\pi}{3}$$

$$V_2 = \pi \int_{-3}^{1} \left(\frac{1}{2} x^2\right)^2 dx = \frac{\pi}{4} \int_{-3}^{1} x^4 dx = \frac{\pi}{4} \cdot \frac{1}{5} x^5 \Big|_{-3}^{1} = \frac{\pi}{20} (1 + 243) =$$

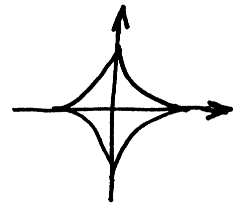
$$= \frac{61}{5} \pi$$

$$V = V_1 - V_2 = \frac{91\pi}{3} - \frac{61\pi}{5} = \frac{272\pi}{15} = 18 \frac{2}{15} \pi$$

tražena zapremina

(#) Izračunati zapreminu tijela koje nastaje rotacijom krive $x = a\cos^3 t$, $y = a\sin^3 t$ oko x-ose.

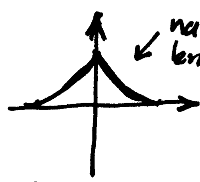
Rj. Data kriva $c: \begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \\ t \in (0, 2\pi] \end{cases}$ je poznata pod imenom astroida i njen grafik je



.

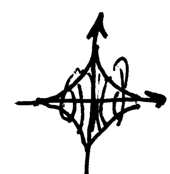
Primjetimo se:

$$V_x = \pi \int_{t_1}^{t_2} [\mu(t)]^2 \, |\eta'(t)| \, dt$$ zapremina tijela kada kriva c rotira oko x-ose, gdje je $c: \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{cases}$



$x = a\cos^3 t \qquad y^2 = a^2 \sin^6 t$

$x' = 3a\cos^2 t \cdot (-\sin t) = -3a \sin t \cos^2 t$



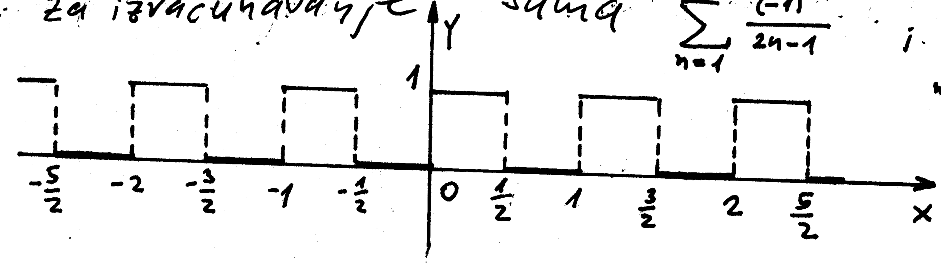
$$V_x = \pi \int_0^{\pi} a^2 \sin^6 t \cdot 3a\, |\sin t|\, |\cos^2 t| \, dt =$$ | kako je figura simetrična u odnosu na y-osu i $\sin t > 0$ za $t \in (0, \frac{\pi}{2})$ |

$$= 2\pi \int_0^{\pi/2} a^2 \sin^6 t \cdot 3a \cdot \sin t \cdot \cos^2 t \, dt = 6a^3\pi \int_0^{\pi/2} \sin^7 t \cos^2 t \, dt$$

$$= 6a^3\pi \int_0^{\pi/2} \underbrace{\sin^6 t}_{(\sin^2 t)^3} \cos^2 t \sin t \, dt = 6a^3\pi \int_0^{\pi/2} \underbrace{(1-\cos^2 t)^3}_{1 - 3\cos^2 t + 3\cos^4 t - \cos^6 t} \cos^2 t \, (-1) \, d\cos t =$$

$$= -6a^2\pi \int_0^{\pi/2} (\cos^2 t - 3\cos^4 t + 3\cos^6 t - \cos^8 t) \, d\cos t = \underset{\text{VJEŽBU}}{\text{ZA}} = \frac{32}{105}\pi a^3$$

Pretvoriti u Furijeov red f-ju definisanu grafikom. Iskoristiti dobijeni rezultat za izračunavanje suma $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ i $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2n-1}$.



R: F-ju predstavljenu grafikom označićemo sa $y=f(x)$.
F-ja je periodična perioda 1, što znači f-ju je dovoljno pretvoriti u Furijeov red u intervalu $[0,1]$.
Furijeovi koeficijenti na intervalu $[a,b]$ se računaju po formuli:

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx, \quad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx \quad i \quad b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$$

Interval $[0,1]$ nije simetričan u odnosu na 0, pa parnost i neparnost ne igraju nikakvu ulogu.
Furijeov red f-je $f(x)$ je oblika $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a} + b_n\sin\frac{2n\pi x}{b-a}\right)$, $x\in[a,b]$

U našem slučaju:

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 2\cdot\frac{1}{2} = 1$$

$$a_n = 2\int_0^1 f(x)\cos 2n\pi x\,dx = 2\int_0^{1/2}\cos 2n\pi x\,dx = 2\,\frac{1}{2n\pi}\sin 2n\pi x\Big|_0^{\frac{1}{2}} = \frac{1}{n\pi}\sin n\pi = 0$$

$$b_n = 2\int_0^1 f(x)\sin 2n\pi x\,dx = 2\int_0^{\frac{1}{2}}\sin 2n\pi x\,dx = 2\,\frac{1}{2n\pi}\left(-\cos 2n\pi x\Big|_0^{\frac{1}{2}}\right) = \frac{(-1)}{n\pi}(\cos n\pi - 1)$$

$$= \frac{(-1)}{n\pi}\left((-1)^n - 1\right) = \frac{1+(-1)^{n+1}}{n\pi}$$

$\boxed{2(2n-1)\pi x}$

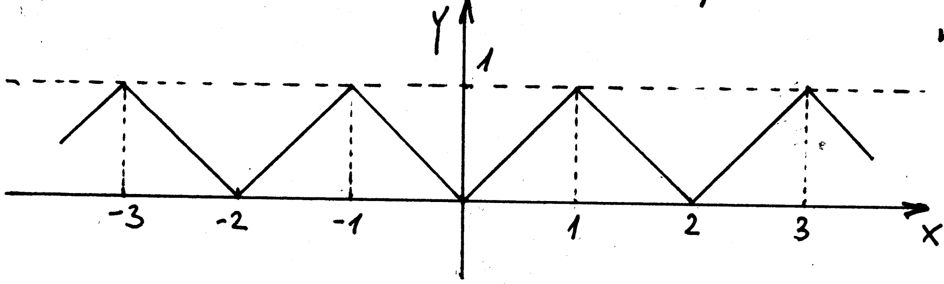
$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n\pi}\sin 2n\pi x = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{(2n-1)\pi}\sin 2(2n-1)\pi x$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2(2n-1)\pi x}{(2n-1)}$$ f-ja razložena u Fourierov red

$f\left(\frac{1}{4}\right) = 1$ (iz grafika), $\sin 2(2n-1)\pi\cdot\frac{1}{4} = \sin(2n-1)\frac{\pi}{2} = (-1)^{n+1}$ pa je $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} = -\frac{\pi}{4}$ (ovaj rezultat se može dobiti na dva načina: U Furijeov red uvrstiti tačku $x=\frac{3}{4}$ ili prethodnu sumu pomnožiti sa (-1)).

(#) F-ju definisanu grafikom razviti u Fourierov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.



Rj: Sa grafika možemo primetiti da je f-ja parna i periodična perioda 2. F-ju je dovoljno razviti u Fourierov red u intervalu $[-1,1]$, pa kako je f-ja parna imaćemo da su $b_n = 0 \ \forall n$.

Ako f-ju označim sa $f(x)$ na intervalu $[-1,1]$ imamo
$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ -x, & -1 \leq x \leq 0 \end{cases}$$

Ako je $f(x)$ integrabilna f-ja na intervalu $[-l, l]$ Fourierove koeficijente računamo po formulama

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Fourierov red f-je $f(x)$ je tad oblika:
$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

F-ja je parna:
$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = 2\int_0^1 x\,dx = 2\cdot\frac{1}{2}x^2\Big|_0^1 = 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = 2\int_0^1 x\cos n\pi x\,dx = \begin{vmatrix} u = x & dv = \cos n\pi x\,dx \\ du = dx & v = \frac{1}{n\pi}\sin n\pi x \end{vmatrix} =$$

$$= \underbrace{\frac{2}{n\pi}x\sin n\pi x\Big|_0^1}_{=0} - \frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx = -\frac{2}{n\pi}\cdot\frac{(-1)}{n\pi}\cos n\pi x\Big|_0^1 = 2\frac{\cos n\pi - \cos 0}{n^2\pi^2}$$

$$a_n = 2\frac{(-1)^n - 1}{n^2\pi^2}, \quad b_n = 0 \ \forall n \Rightarrow f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{\pi^2}\cdot\frac{-4}{(2n-1)^2}\cos(2n+1)\pi x$$

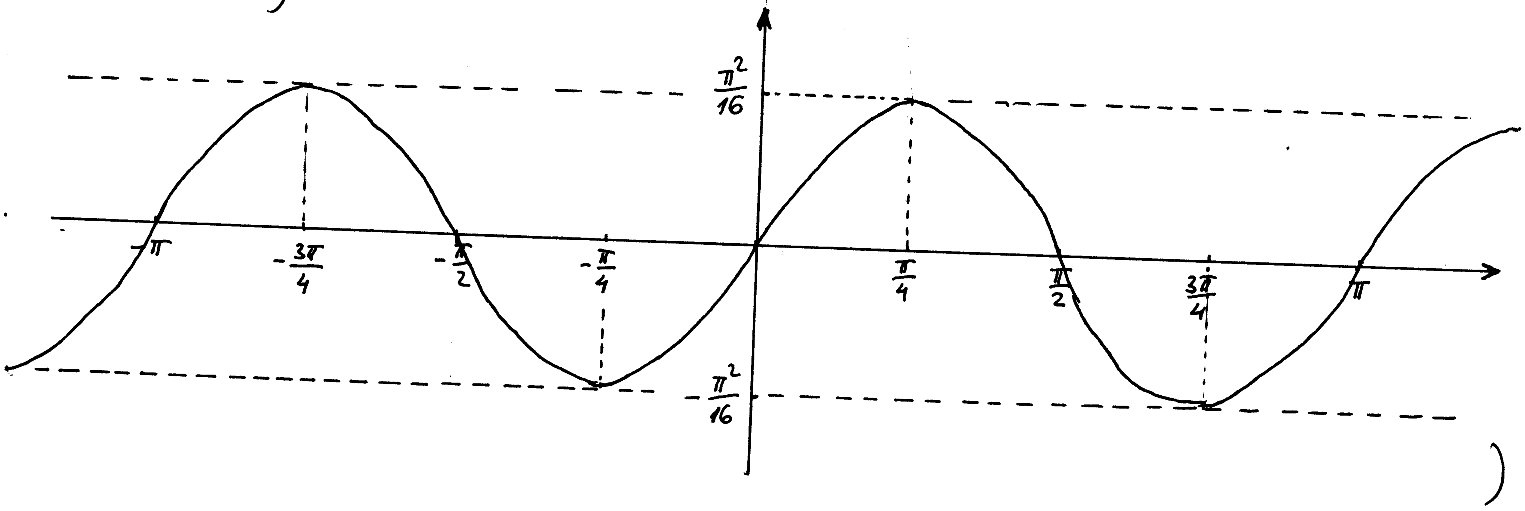
$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n+1)^2}$$ razlaganje f-je u Fourierov red

Za $x=0$, $f(0)=0 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

(#) Razviti f-ju $f(x)=x\left(\frac{\pi}{2}-x\right)$ po sinusima višestrukih uglova u intervalu $\left(0,\frac{\pi}{2}\right)$.

Rj.

F-ja koju razvijamo u red po sinusima grafički izgleda ovako:



Neka je $f(x)$ neka 2π periodična f-ja. Tada

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$ Furijeov red f-je $f(x)$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$ Furijeovi koeficijenti f-je $f(x)$

Ako je $f(x)$ parna f-ja tada $f(x)\sin nx$ je neparna $\Rightarrow b_n=0\ \forall n\in\mathbb{Z}$

Ako je $f(x)$ neparna f-ja tada $f(x)\cos nx$ neparna $\Rightarrow a_n=0\ \forall n\in\mathbb{Z}$

Trebamo napraviti neparno produženje f-je $f(x)$ (novu f-ju ću nazvati $f^*(x)$)

$$f^*(x)=\begin{cases}f(x), & x\in\left(0,\frac{\pi}{2}\right)\\ -f(-x), & x\in\left(-\frac{\pi}{2},0\right)\end{cases}=\begin{cases}x\left(\frac{\pi}{2}-x\right), & x\in\left(0,\frac{\pi}{2}\right)\\ x\left(\frac{\pi}{2}+x\right), & x\in\left(-\frac{\pi}{2},0\right)\end{cases}$$

Izračunajmo Furijeove koeficijente

$$b_n=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f^*(x)\sin nx\,dx \ \underset{\text{f-ja}}{\overset{f^*(x)\sin nx \text{ parna}}{=}}\ \frac{2}{\pi}\int_0^{\frac{\pi}{2}}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}x\left(\frac{\pi}{2}-x\right)\sin nx\,dx=$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2}x-x^2\right)\sin nx\,dx=\frac{2}{\pi}\left(\frac{\pi}{2}\int_0^{\pi/2}x\sin nx\,dx-\int_0^{\pi/2}x^2\sin nx\,dx\right)\overset{(*)}{=}$$

$$I_1 = \int_0^{\pi/2} x \sin nx\, dx = \begin{vmatrix} u=x & dv=\sin nx\, dx \\ du=dx & v=-\frac{1}{n}\cos nx \end{vmatrix} = -\frac{1}{n} x \cos nx \Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\pi/2} \cos nx\, dx =$$

$$= -\frac{1}{n}\left(\frac{\pi}{2}\cos n\cdot\frac{\pi}{2} - 0\right) + \frac{1}{n}\cdot\frac{1}{n}\sin nx\Big|_0^{\frac{\pi}{2}} = -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}$$

($\sin\frac{n\pi}{2}$ i $\cos\frac{n\pi}{2}$ mogu uzimati tri vrijednosti $0, 1, -1$)

$$I_2 = \int_0^{\pi/2} x^2 \sin nx\, dx = \begin{vmatrix} u=x^2 & dv=\sin nx\, dx \\ du=2x\, dx & v=-\frac{1}{n}\cos nx \end{vmatrix} = -\frac{1}{n}x^2\cos nx\Big|_0^{\frac{\pi}{2}} +$$

$$+\frac{2}{n}\int_0^{\pi/2} x\cos nx\, dx = \begin{vmatrix} u=x & dv=\cos nx\, dx \\ du=dx & v=\frac{1}{n}\sin nx \end{vmatrix} = -\frac{\pi^2}{4n}\cos\frac{n\pi}{2} +$$

$$+\frac{2}{n}\Bigg(\underbrace{\frac{1}{n}x\sin nx\Big|_0^{\frac{\pi}{2}}}_{\frac{\pi}{2}\sin\frac{n\pi}{2}} - \frac{1}{n}\underbrace{\int_0^{\pi/2}\sin nx\, dx}_{\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\frac{\pi}{2}}}\Bigg) = \underline{\frac{-\pi^2}{4n}\cos\frac{n\pi}{2}} + \frac{\pi}{n^2}\sin\frac{n\pi}{2} + \underline{\frac{2}{n^3}\left(\cos\frac{n\pi}{2}-1\right)}$$

$$= \left(\frac{2}{n^3} - \frac{\pi^2}{4n}\right)\cos\frac{n\pi}{2} + \frac{\pi}{n^2}\sin\frac{n\pi}{2} - \frac{2}{n^3}$$

Primjetimo

$$I_1 = \begin{cases} -\frac{\pi}{2n}(-1)^k, & n=2k \\ \frac{1}{n^2}(-1)^k, & n=2k+1 \end{cases} \quad k=0,1,2,\ldots \qquad I_2 = \begin{cases} \left(\frac{2}{n^3}-\frac{\pi^2}{4n}\right)(-1)^k - \frac{2}{n^3}, & n=2k \\ \frac{\pi}{n^2}(-1)^k - \frac{2}{n^3}, & n=2k+1 \end{cases} \quad k=0,1,2,\ldots$$

$$\underline{\underline{(*)}} -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{2}{\pi}\left(\left(\frac{2}{n^3}-\frac{\pi^2}{4n}\right)\cos\frac{n\pi}{2} + \frac{\pi}{n^2}\sin\frac{n\pi}{2} - \frac{2}{n^3}\right) =$$

$$= \underline{-\frac{\pi}{2n}\cos\frac{n\pi}{2}} + \underline{\underline{\frac{1}{n^2}\sin\frac{n\pi}{2}}} + \left(\frac{-4}{\pi n^3} + \underline{\frac{\pi}{2n}}\right)\underline{\cos\frac{n\pi}{2}} - \underline{\underline{\frac{2}{n^2}\sin\frac{n\pi}{2}}} + \frac{4}{\pi n^3} =$$

$$= -\frac{4}{\pi n^3}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{4}{\pi n^3} = \begin{cases} -\frac{4}{\pi n^3}(-1)^k + \frac{4}{\pi n^3}, & n=2k \\ -\frac{1}{n^2}(-1)^k + \frac{4}{\pi n^3}, & n=2k+1 \end{cases} \quad k=0,1,2,3,\ldots$$

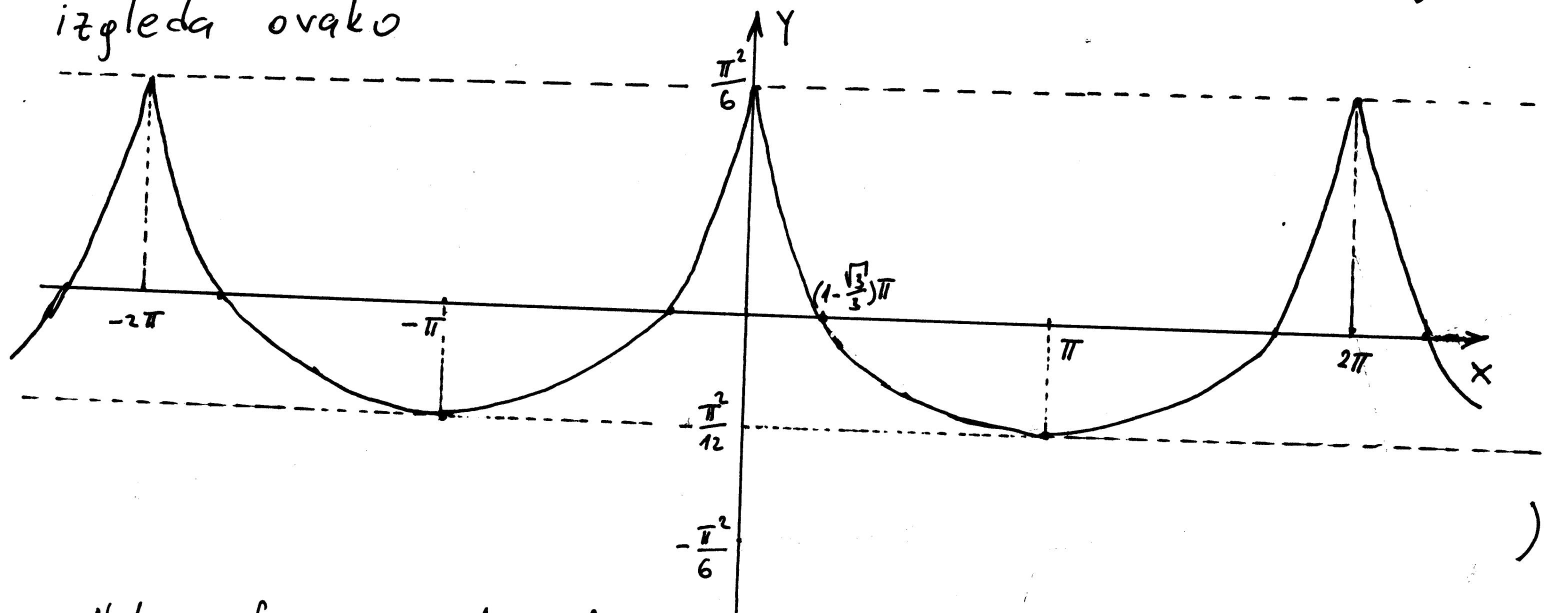
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx =$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\left(\frac{(-1)^{k+1}+1}{(2k)^3}\right)\sin nx + \sum_{k=0}^{\infty}\left(\frac{(-1)^{k+1}}{(2k+1)^2} + \frac{4}{\pi(2k+1)^3}\right)\sin nx$$

razvoj f-je po sinusima

Razviti f-ju $f(x) = \frac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0, \pi)$.

Rj. F-ju koju razvijamo u Furijeov red po kosinusima grafički izgleda ovako



Neka je $f(x)$ 2π periodična f-ja.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \text{Furijeov red f-je } f(x)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{Furijeovi koeficijenti f-je } f(x)$$

Ako je $f(x)$ parna tada je $f(x)\sin nx$ neparna $\Rightarrow$ $b_n = 0$ $\forall n \in \mathbb{Z}$

Ako je $f(x)$ neparna tada je $f(x)\cos nx$ neparna $\Rightarrow$ $a_n = 0$ $\forall n \in \mathbb{Z}$

Trebamo napraviti parno produženje f-je $f(x)$ (novu f-ju nazovimo $f^*(x)$)

$$f^*(x) = \begin{cases} f(x), & x \in (0,\pi) \\ f(-x), & x \in (-\pi, 0) \end{cases} = \begin{cases} \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2), & x \in (0,\pi) \\ \frac{1}{12}(3x^2 + 6\pi x + 2\pi), & x \in (-\pi, 0) \end{cases}$$

Izračunajmo Fourijeove koeficijente

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\,dx \overset{f^* \text{ parna}}{=\!=\!=} \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\,dx = \frac{1}{6\pi}\Big(3\cdot\frac{1}{3}x^3\Big|_0^{\pi} -$$

$$-6\pi\cdot\frac{1}{2}x^2\Big|_0^{\pi} + 2\pi^2\cdot x\Big|_0^{\pi}\Big) = \frac{1}{6\pi}(\pi^3 - 3\pi^3 + 2\pi^3) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\cos nx\,dx \overset{f^*(x) \text{ parna}}{=\!=\!=} \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\cos nx\,dx =$$

$$= \frac{1}{6\pi}\left(3\int_0^{\pi} x^2 \cos nx\,dx - 6\pi\int_0^{\pi} x\cos nx\,dx + 2\pi^2\int_0^{\pi}\cos nx\,dx\right) \overset{(*)}{=}$$

$$I_1 = \int_0^{\pi} x^2\cos nx\,dx = \begin{vmatrix} u = x^2 & dv = \cos nx\,dx \\ du = 2x\,dx & v = \frac{1}{n}\sin nx \end{vmatrix} = \frac{1}{n}x^2 \sin nx\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx =$$

$$= \begin{vmatrix} u = x & dv = \sin nx\,dx \\ du = dx & v = -\frac{1}{n}\cos nx \end{vmatrix} = \frac{1}{n}\underbrace{(\pi^2 \sin n\pi - 0)}_{=0} - \frac{2}{n}\left(-\frac{1}{n}x\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)$$

$$= \frac{2}{n^2}(\pi\cos n\pi - 0) - \frac{2}{n^2}\sin nx\Big|_0^{\pi} = (-1)^n\frac{2\pi}{n^2}$$

$$I_2 = \int_0^{\pi} x\cos nx\,dx = \begin{vmatrix} u = x & dv = \cos nx\,dx \\ du = dx & v = \frac{1}{n}\sin nx \end{vmatrix} = \frac{1}{n}x\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx =$$

$$= \frac{1}{n}\underbrace{(\pi\sin n\pi - 0)}_{=0} - \frac{1}{n}\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\pi} = \frac{1}{n^2}(\cos n\pi - \cos 0) = \frac{1}{n^2}((-1)^n - 1)$$

$$I_3 = \int_0^{\pi}\cos nx\,dx = \frac{1}{n}\sin nx\Big|_0^{\pi} = \frac{1}{n}(\sin n\pi - \sin 0) = 0$$

$$\overset{(*)}{=} \frac{1}{2\pi}(-1)^n\frac{2\pi}{n^2} - \frac{1}{n^2}((-1)^n - 1) = \frac{1}{n^2}((-1)^n - (-1)^n + 1) = \frac{1}{n^2}$$

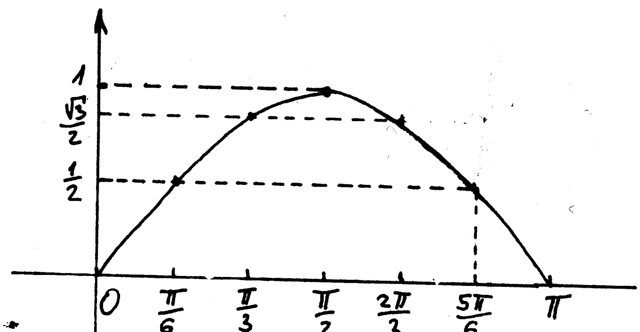
$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx, \quad x\in(0,\pi)$$

razvoj f-je f(x) u red po kosinusima

(Primjetimo da dobijeni rezultat možemo iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Naime ako stavimo $x=0$ imamo

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$).

Dio grafika f-je $y=f(x)$ je prikazan na slici.



Datu f-ju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2}$.

Rj. Furijerov red za f-ju $y=f(x)$ na intervalu (a,b) glasi

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right) \quad \ldots (1)$$

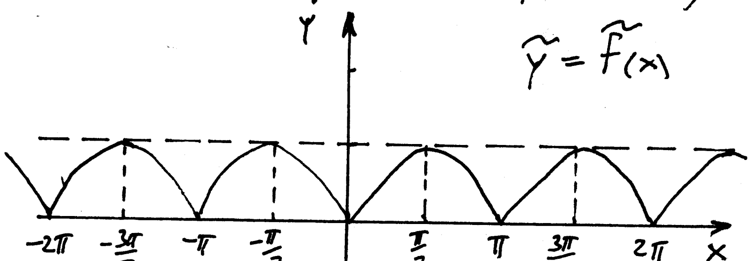
gdje se Furijer-ovi koeficijenti računaju po formuli

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx, \quad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx, \quad b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx \quad \ldots (2)$$

Prema formuli (1) da bi f-ju pretvorili u Furijer-ov red samo po cos-inusima, potrebno je i dovoljno imati f-ju za koju će vrijediti da je $b_n = 0$. Prema formuli (2) da bi b_n bio jednak nuli, interval (a,b) mora biti simetričan u odnosu na nulu i f-ja $f(x)$ mora biti parna (zato što

$$b_n = \frac{2}{b-a}\int_a^b \underbrace{\underbrace{f(x)}_{\text{parna}} \underbrace{\sin\frac{2n\pi x}{b-a}}_{\text{neparna}}}_{\text{neparna}}\,dx \Big).$$

Prema tome pravimo proširenje date f-je:



Naravno f-ja koju pretvaramo u Furijer-ov red mora biti periodična.

Prvo primjetimo da je data f-ja $y=f(x)$ u stvari f-ja $y=\sin x$ na intervalu $(0,\pi)$. Proširenje date f-je je u stvari f-ja $\tilde{y}=|\sin x|$.

$(a,b)=(-\pi,\pi)$, $\quad \frac{2}{b-a}=\frac{2}{\pi-(-\pi)}=\frac{2}{2\pi}=\frac{1}{\pi}$, $\quad \frac{2n\pi x}{b-a}=\frac{2n\pi x}{2\pi}=nx$

F-ja $\tilde{y}=|\sin x|$ je parna $\Rightarrow$ $b_n=0$.

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\tilde{f}(x)\cos nx}_{\text{parna f-ja}}\,dx=\frac{2}{\pi}\int_0^{\pi}\tilde{f}(x)\cos nx\,dx=$$

$$=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\left(\sin(n+1)x+\sin(n-1)x\right)dx=$$

$$\begin{aligned}\sin(A+B)&=\sin A\cos B+\sin B\cos A\\ +\ \sin(A-B)&=\sin A\cos B-\sin B\cos A\\ \hline \sin A\cos B&=\tfrac{1}{2}\left(\sin(A+B)+\sin(A-B)\right)\end{aligned}$$

$$=\frac{1}{\pi}\left(\frac{-1}{1+n}\cos(1+n)x\Big|_0^{\pi}+\frac{-1}{1-n}\cos(1-n)x\Big|_0^{\pi}\right)=$$

$$=\frac{1}{\pi}\left(\frac{-1}{1+n}\Big(\underbrace{\cos(1+n)\pi}_{(-1)^{n+1}}-1\Big)+\frac{-1}{1-n}\Big(\underbrace{\cos(1-n)\pi}_{(-1)^{n-1}}-1\Big)\right)=$$

$$=\frac{1}{\pi}\left(\frac{(-1)^n+1}{1+n}+\frac{(-1)^n+1}{1-n}\right)=\frac{1}{\pi}\,\frac{((-1)^n+1)(1+n+1-n)}{(1+n)(1-n)}=\frac{1}{\pi}\,\frac{2}{(1+n)(1-n)}\left((-1)^n+1\right)$$

za $n=1,2,3,\dots$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\tilde{f}(x)\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=\frac{2}{\pi}\left(-\cos x\Big|_0^{\pi}\right)=\frac{-2}{\pi}(-1-1)=\frac{4}{\pi}$$

$$\tilde{f}(x)\sim\frac{4}{\pi}+\sum_{n=1}^{\infty}\frac{2}{\pi}\,\frac{(-1)^n+1}{1-n^2}\cos nx=\frac{4}{\pi}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{1-4k^2}$$

Prema tome $\sin x\sim\frac{4}{\pi}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{1-4k^2}$, $\quad x\in(0,\pi)$

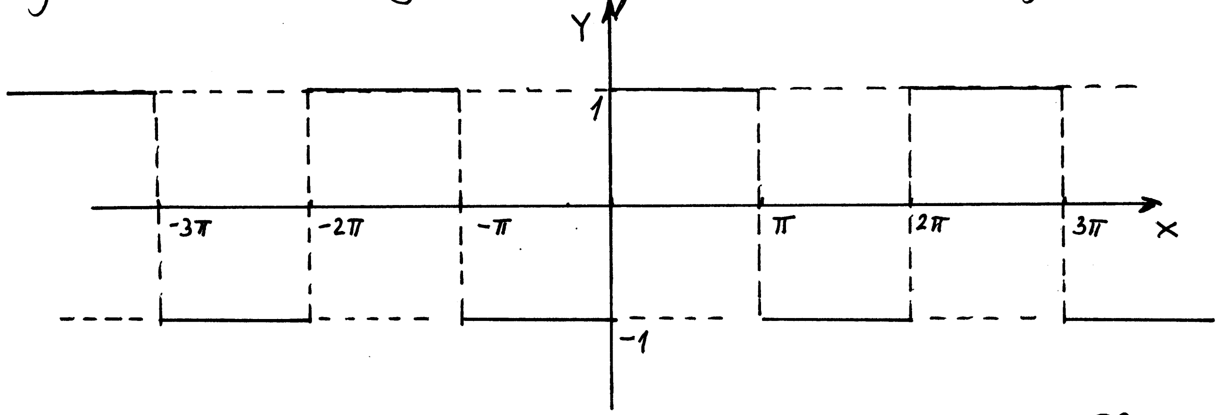
Za $x=\frac{\pi}{2}$ imamo:

$$\sin\frac{\pi}{2}=\frac{4}{\pi}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2k\cdot\frac{\pi}{2}}{1-4k^2}\Rightarrow 1-\frac{4}{\pi}=\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos k\pi}{1-4k^2}\quad /\cdot\pi$$

$$4\sum_{k=1}^{\infty}\frac{(-1)^k}{1-4k^2}=\pi-4$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}=\frac{\pi}{4}-1\quad \text{tražena suma}$$

(#) F-ju definisanu grafikom pretvoriti u Furijer-ov red.



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Rj: Primjetimo da je data f-ja periodična, period 2π, pa je možemo pretvoriti u Furijer-ov red. Kada je x-osa data u radijanima Furijer-ov red izgleda $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

gdje se Furijer-ovi koeficijenti računaju u obliku (za 2π per. f-ju)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{2}\int_{0}^{\pi} 1\,dx = \left(-\frac{1}{\pi}\right)x\Big|_{-\pi}^{0} + \frac{1}{\pi}x\Big|_{0}^{\pi} = -1+1=0$$

(iznad: $0-(-\pi)$)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx = \left(-\frac{1}{\pi}\right)\frac{1}{n}\sin nx\Big|_{-\pi}^{0} + \frac{1}{\pi}\cdot\frac{1}{n}\sin nx\Big|_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} b_n \sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx = \left(-\frac{1}{\pi}\right)\left(-\frac{1}{n}\right)\cos nx\Big|_{-\pi}^{0} + \frac{1}{\pi}\left(-\frac{1}{n}\right)\cos nx\Big|_{0}^{\pi}$$

$$= \frac{1}{n\pi}(1-\cos n\pi) - \frac{1}{n\pi}(\cos n\pi - 1) = \frac{2}{n\pi}(1-\cos n\pi)$$

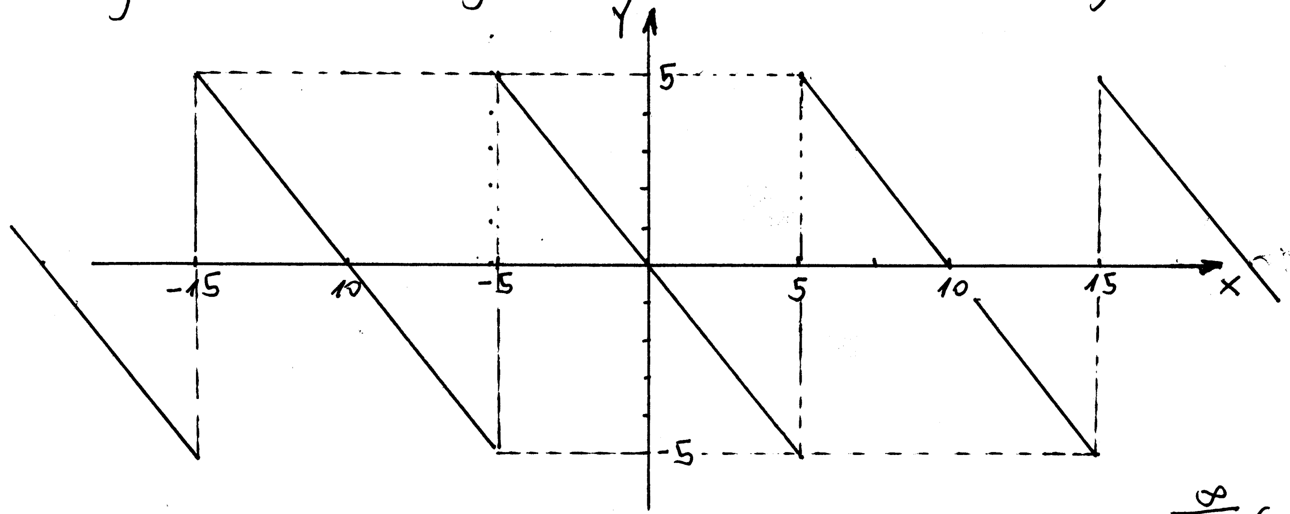
$$1-\cos n\pi = 1-(-1)^n = \begin{cases} 0, & n=2k \\ 2, & n=2k+1 \end{cases} \quad k=0,1,2,\ldots$$

$$\boxed{\sin(2k+1)\frac{\pi}{2} = (-1)^k}$$

$$f(x) \sim \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1} \quad \text{traženi Furijerov red}$$

$$f\left(\frac{\pi}{2}\right)=1 \quad \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\frac{\pi}{2}}{2k+1} = 1 \implies \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4} \quad \text{tražena suma}$$

(#) Funkciju definisanu grafikom pretvoriti u Furijeov red.



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi}{50}$.

Rj. Primjetimo da je data f-ja periodična perioda 10. Prema tome dovoljno ju je pretvoriti u Furijeov red na proizvoljnom intervalu perioda 10. Pa posmatrajmo npr. interval $[-5,5]$. F-ja na ovom intervalu ima oblik $f(x) = -x$.

Furijeov red f-je $f(x)$ na intervalu $[a,b]$ ima oblik:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$

gdje su $a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$, $a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx$ i $b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$, $n = 1, 2, \dots$

Furijeovi koeficijenti. U našem slučaju interval $[a,b]$ je $[-5,5]$ pa je $b-a = 5+5 = 10$, $\frac{2}{10} = \frac{1}{5}$, $\frac{2n\pi x}{b-a} = \frac{2n\pi x}{10} = \frac{n\pi x}{5}$.

$$a_0 = \frac{1}{5}\int_{-5}^{5}(-x)\,dx = \frac{1}{5}(-1)\cdot\frac{1}{2}x^2\Big|_{-5}^{5} = 0$$

$d\left(\frac{n\pi x}{5}\right) = \frac{n\pi}{5}dx$

$$a_n = \frac{1}{5}\int_{-5}^{5}(-x)\cos\frac{n\pi x}{5}\,dx = \begin{vmatrix} u = x & dv = \cos\frac{n\pi x}{5}dx \\ du = dx & v = \frac{5}{n\pi}\sin\frac{n\pi x}{5}\end{vmatrix} =$$

$$= -\frac{1}{5}\Bigg(\overbrace{\frac{5}{n\pi}x\sin\frac{n\pi x}{5}\Big|_{-5}^{5}}^{=0} - \frac{5}{n\pi}\int_{-5}^{5}\sin\frac{n\pi x}{5}\,dx = \frac{1}{n\pi}\left(-\frac{5}{n\pi}\right)\cos\frac{n\pi x}{5}\Big|_{-5}^{5} = 0$$

$$b_n = \frac{1}{5}\int_{-5}^{5}(-x)\sin\frac{n\pi x}{5}\,dx = \left| \begin{matrix} u = x & dv = \sin\frac{n\pi x}{5}\,dx \\ du = dx & v = \frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right) \end{matrix} \right| =$$

$$= -\frac{1}{5}\left(\frac{-5}{n\pi}\,x\cos\frac{n\pi x}{5}\Big|_{-5}^{5} + \frac{5}{n\pi}\int_{-5}^{5}\cos\frac{n\pi x}{5}\,dx\right) =$$

$$= \frac{1}{n\pi}\left(5\cos n\pi - (-5)\cos n\pi\right) - \underbrace{\frac{1}{n\pi}\cdot\frac{5}{n\pi}\sin\frac{n\pi x}{5}\Big|_{-5}^{5}}_{=0} =$$

$$= \frac{10}{n\pi}\cos n\pi = \frac{10}{n\pi}(-1)^n$$

Prema tome

$$-x \sim \sum_{n=1}^{\infty}\frac{10}{n\pi}(-1)^n\sin\frac{n\pi x}{5} \quad \text{za } \forall x\in[-5,5]$$

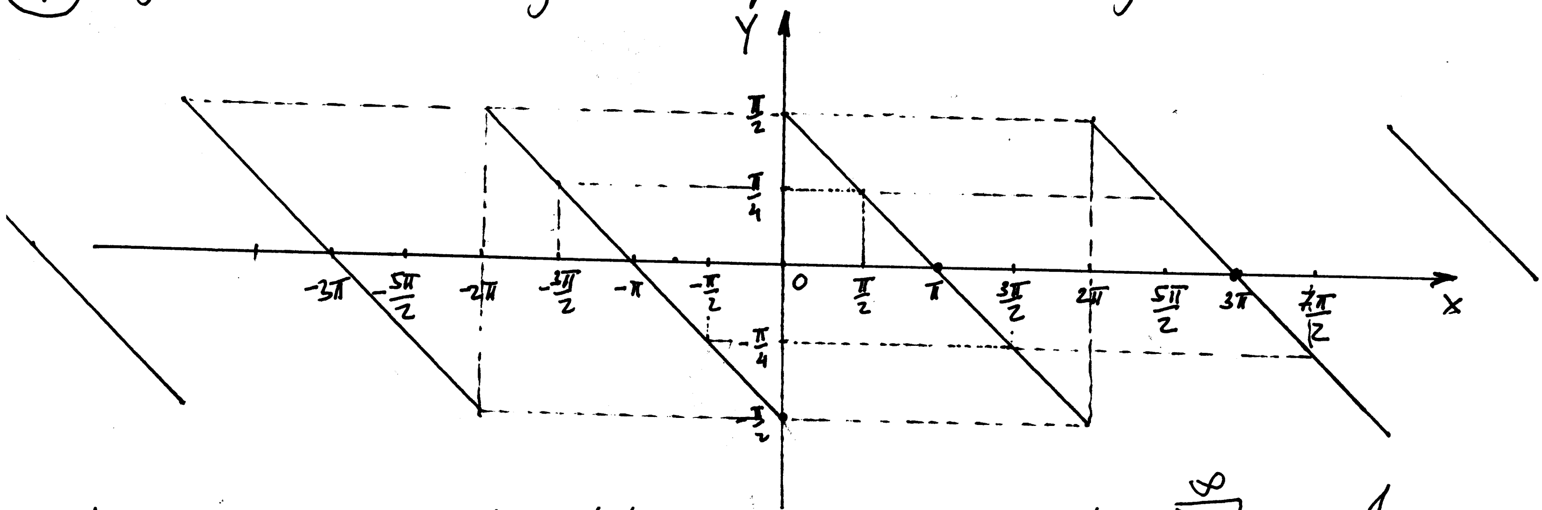
tj.
$$-x = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{5} \quad \text{za } \forall x\in[-5,5]$$

Ako za x uzmemo $x = \frac{1}{10}$ imamo:

$$-\frac{1}{10} = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi\cdot\frac{1}{10}}{5}$$

tj.
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{50} = -\frac{\pi}{100} \quad \text{tražena suma}$$

(#) F-ju definisanu grafikom pretvoriti u Furijeov red



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{1}{(4n-1)(4n-3)}$.

Rj. Prvo primjetimo da je f-ja periodična što znači da se može pretvoriti u Furijeov red. Dalje, primjetimo da je period 2π što znači da možemo posmatrati npr. interval $[0, 2\pi]$.

F-ja na intervalu $[0, 2\pi]$ prolazi kroz sljedeće tačke $(0, \frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{\pi}{4})$ $(\pi, 0)$, $(\frac{3\pi}{2}, -\frac{\pi}{4})$, $(2\pi, -\frac{\pi}{2})$. Jednačina prave kroz dvije tačke je

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \quad \xrightarrow{\text{ako posmatramo } \underset{x_1\ y_1}{(0,\frac{\pi}{2})} \text{ i } \underset{x_2\ y_2}{(\pi,0)}} \quad \frac{x-0}{\pi} = \frac{y-\frac{\pi}{2}}{-\frac{\pi}{2}}$$

$$\Rightarrow \quad y - \frac{\pi}{2} = \frac{x}{\pi}\cdot\left(-\frac{\pi}{2}\right) \Rightarrow y - \frac{\pi}{2} = -\frac{x}{2} \Rightarrow y = \frac{\pi - x}{2}.$$

Furijeov red na intervalu $[a, b]$ je oblika

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$

gdje se Furijeovi koeficijenti računaju po formuli $\qquad \frac{2n\pi x}{2\pi} = nx$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx, \qquad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi}{b-a}\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\,dx = \frac{1}{2\pi}\left(\pi x\Big|_0^{2\pi} - \frac{1}{2}x^2\Big|_0^{2\pi}\right) = \frac{1}{2\pi}(2\pi^2 - 2\pi^2) = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\cos nx\,dx = \begin{vmatrix} u = \pi - x & dv = \cos nx\,dx \\ du = -dx & v = \frac{1}{n}\sin nx \end{vmatrix} =$$

$$=\frac{1}{2\pi}\left(\frac{1}{n}(\pi-x)\sin nx\Big|_0^{2\pi}+\frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right)=\underbrace{-\frac{1}{2n\pi}\sin 2n\pi}_{=0}+\frac{1}{2n\pi}\cdot\frac{-1}{n}\cos nx\Big|_0^{2\pi}=$$

$$=-\frac{1}{2n^2\pi}\left(\cos 2n\pi-1\right)=\frac{1}{2n^2\pi}\left(1-\cos 2n\pi\right)=\frac{1}{2n^2\pi}(1-1)=0$$

Sa grafika date f-je možemo primjetiti da je f-ja simetrična u odnosu na koordinatni početak tj. da je neparna, pa je $a_0=0$ i $a_n=0$ $\forall n\in\mathbb{N}$.

$$b_n=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\sin nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\sin nx\,dx=\begin{vmatrix}u=\pi-x & dv=\sin nx\,dx\\ du=-dx & v=-\frac{1}{n}\cos nx\end{vmatrix}$$

$$=\frac{1}{2\pi}\left(-\frac{1}{n}(\pi-x)\cos nx\Big|_0^{2\pi}-\frac{1}{n}\int_0^{2\pi}\cos nx\,dx\right)=-\frac{1}{2n\pi}\left(-\pi\cos 2n\pi-\pi\right)-$$

$$-\frac{1}{2n\pi}\sin nx\Big|_0^{2\pi}=\frac{1}{2n}\left(\cos 2n\pi+1\right)=\frac{1}{2n}\cdot 2=\frac{1}{n}$$

Prema tome

$$f(x)\sim\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Ako za x uzmemo $\frac{\pi}{2}$ imamo

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi}{2}n=\sum_{k=1}^{\infty}\Big(\frac{1}{4k-1}\sin(4k-1)\frac{\pi}{2}+\frac{1}{4k-2}\underbrace{\sin(4k-2)\frac{\pi}{2}}_{\sin(2k-1)\pi=0}+$$

$$+\frac{1}{4k-3}\sin(4k-3)\frac{\pi}{2}+\frac{1}{4k}\underbrace{\sin 4k\frac{\pi}{2}}_{=\sin 2k\pi=0}\Big)=$$

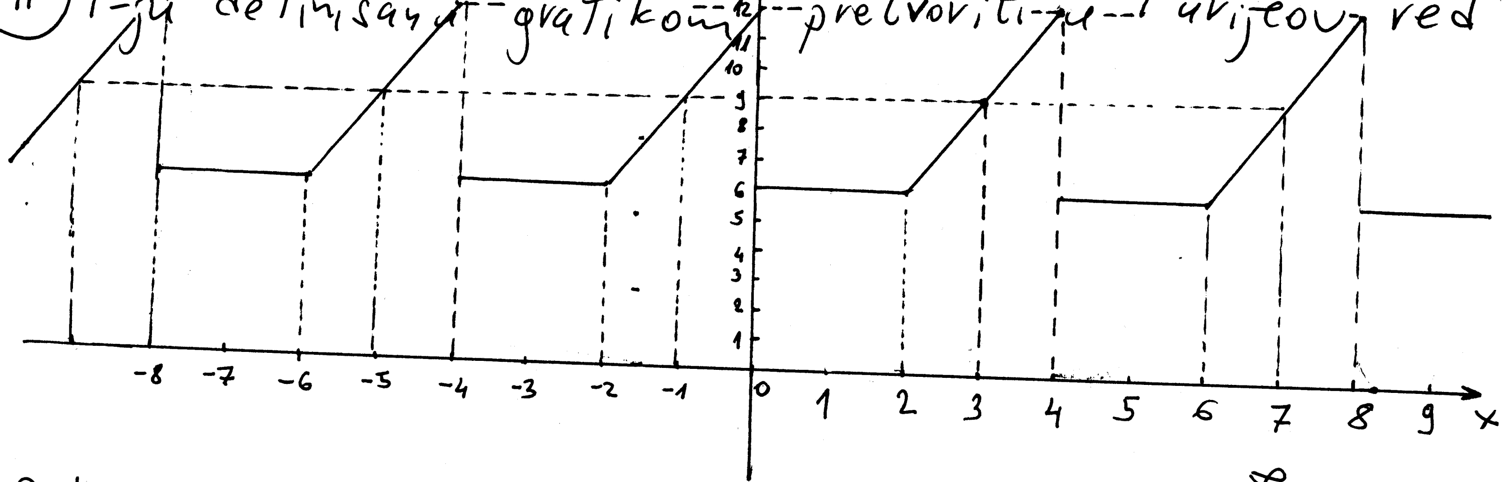
$$=\sum_{k=1}^{\infty}\frac{1}{4k-1}\sin\underbrace{(4k-1)\frac{\pi}{2}}_{=3,7,11,15}+\sum_{k=1}^{\infty}\frac{1}{4k-3}\sin\underbrace{(4k-3)\frac{\pi}{2}}_{=1,5,9,13}$$

$$=-1 \qquad\qquad =1$$

$$=\sum_{k=1}^{\infty}\left(-\frac{1}{4k-1}+\frac{1}{4k-3}\right)=\sum_{k=1}^{\infty}\frac{4k-1-4k+3}{(4k-1)(4k-3)}=\sum_{k=1}^{\infty}\frac{2}{(4k-1)(4k-3)}$$

Kako je $f(\frac{\pi}{2})=\frac{\pi}{4}$ to je $\sum_{n=1}^{\infty}\frac{2}{(4n-1)(4n-3)}=\frac{\pi}{4}$.

F-ja definisana grafikom pretvoriti u Furijeov red



Dobijeni rezultat iskoristiti za sumiranje reda $\sum\limits_{k=1}^{\infty}\frac{1}{(2k-1)^2}$.

Rj. Prvo primjetimo da je data f-ja periodična perioda 4, što znači da je možemo pretvoriti u Furijeov red i to dovoljno nam je pretvoriti u Furijeov red na intervalu $(0,4)$.
Data f-ja na intervalu $(0,4)$ je definisana na sljedeći način $f(x)=\begin{cases}6, & x\in(0,2]\\ 3x, & x\in(2,4)\end{cases}$.

Furijerov red na proizvoljnom intervalu $[a,b]$ izgleda

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$$

a Furijerovi koeficijenti računaju po formulama

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\qquad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad n=1,2,\dots$$

$$b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx,\quad n=1,2,\dots$$

Što znači Furijerov red na intervalu $[0,4)$ je

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{2}+b_n\sin\frac{n\pi x}{2}\right)$$

Izračunajmo Furijeove koeficijente

$$a_0=\frac{1}{2}\int_0^4 f(x)\,dx=\frac{1}{2}\int_0^2 6\,dx+\frac{1}{2}\int_2^4 3x\,dx=3x\Big|_0^2+\frac{3}{2}\cdot\frac{1}{2}x^2\Big|_2^4=6+\frac{3}{4}\cdot\overbrace{12}^{16-4}=15$$

$$a_n=\frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,dx=\frac{1}{2}\int_0^2 6\cos\frac{n\pi x}{2}\,dx+\frac{1}{2}\int_2^4 3x\cos\frac{n\pi x}{2}\,dx=\left|\begin{matrix}u=x & dv=\cos\frac{n\pi x}{2}dx\\ du=dx & v=\frac{2}{n\pi}\sin\frac{n\pi x}{2}\end{matrix}\right.$$

$$= \underbrace{3 \cdot \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_0^2}_{0-0} + \frac{3}{2}\left[\underbrace{\frac{2}{n\pi} x \sin \frac{n\pi x}{2} \Big|_2^4}_{0-0} - \frac{2}{n\pi} \int_2^4 \sin \frac{n\pi x}{2} dx \right] =$$

$$= -\frac{3}{n\pi} \int_2^4 \sin \frac{n\pi x}{2} dx = \frac{3}{n\pi} \cdot \frac{2}{n\pi} \cos \frac{n\pi x}{2} \Big|_2^4 = \frac{6}{n^2\pi^2} (1 - \cos n\pi), \quad n \neq 0$$

Odavde vidimo $a_n = \begin{cases} 0, & n \text{ parno} \\ \frac{12}{n^2\pi^2}, & n \text{ neparno} \end{cases} \quad n \in \mathbb{N}$

$$b_n = \frac{1}{2} \int_0^4 f(x) \sin \frac{n\pi x}{2} dx = \frac{1}{2} \int_0^2 6 \sin \frac{n\pi x}{2} dx + \frac{1}{2} \int_2^4 3x \sin \frac{n\pi x}{2} dx = \left| \begin{matrix} u = x & dv = \sin \frac{n\pi x}{2} dx \\ du = dx & v = -\frac{2}{n\pi} \cdot \cos \frac{n\pi x}{2} \end{matrix} \right|$$

$$= 3\left(-\frac{2}{n\pi}\right) \cos \frac{n\pi x}{2} \Big|_0^2 + \frac{3}{2}\left[-\frac{2}{n\pi} x \cos \frac{n\pi x}{2} \Big|_2^4 + \frac{2}{n\pi} \int_2^4 \cos \frac{n\pi x}{2} dx \right] =$$

$$= \left(-\frac{6}{n\pi}\right)(\cos n\pi - 1) + \frac{3}{2}\left[\left(-\frac{2}{n\pi}\right)(4\cos 2n\pi - 2\cos n\pi) + \underbrace{\frac{2}{n\pi} \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_2^4}_{0-0} \right]$$

$$= \frac{6}{n\pi}(1 - \cos n\pi) - \frac{3}{n\pi}(4 - 2\cos n\pi) = \frac{6}{n\pi}(1 - \cos n\pi) - \frac{6}{n\pi}(2 - \cos n\pi)$$

$$= \frac{6}{n\pi}(1 - \cos n\pi - 2 + \cos n\pi) = -\frac{6}{n\pi}$$

Prema tome $f(x) \sim \frac{15}{2} + \sum_{n=1}^{\infty} \left(\frac{6}{n^2\pi^2}(1 - \cos n\pi) \cos \frac{n\pi x}{2} + \left(-\frac{6}{n\pi}\right) \sin \frac{n\pi x}{2} \right)$

$$= \frac{15}{2} + \sum_{k=1}^{\infty} \left(\frac{12}{(2k-1)^2\pi^2} \cos \frac{(2k-1)\pi x}{2} - \frac{6}{k\pi} \sin \frac{k\pi x}{2} \right)$$

$$f(x) \sim \frac{15}{2} + \frac{12}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos (2k-1)\frac{\pi}{2} x}{(2k-1)^2} - \frac{6}{\pi} \sum_{k=1}^{\infty} \frac{\sin k\frac{\pi}{2} x}{k}$$

Za $x=2$ imamo

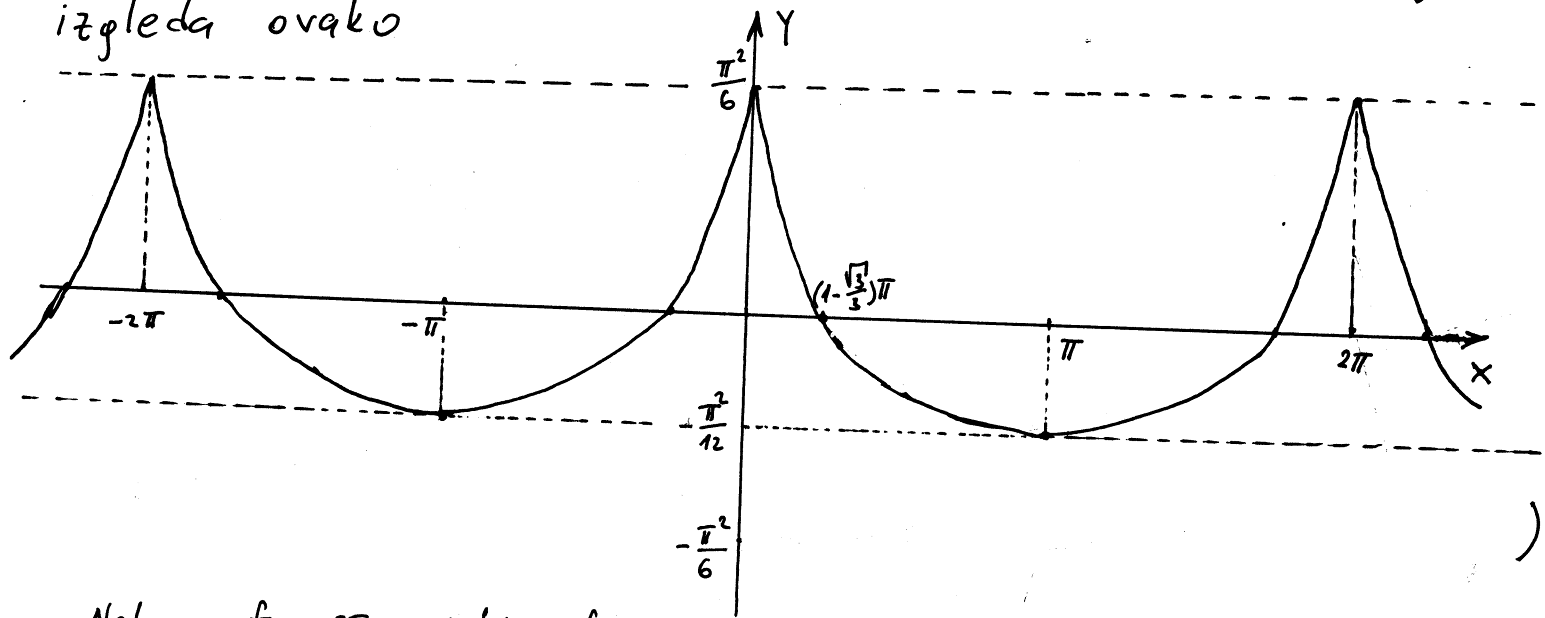
$$f(2) = \frac{15}{2} + \frac{12}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos(2k-1)\pi}{(2k-1)^2} - \underbrace{\frac{6}{\pi} \sum_{k=1}^{\infty} \frac{\sin k\pi}{k}}_{=0}$$

$$6 = \frac{15}{2} + \frac{12}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)}{(2k-1)^2} \Rightarrow -\frac{12}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = -\frac{3}{2}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \left(-\frac{3}{2}\right)\left(-\frac{\pi^2}{12}\right) = \frac{\pi^2}{2 \cdot 4} \qquad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \text{ tražena suma}$$

#) Razviti f-ju $f(x) = \frac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0,\pi)$.

Rj. F-ju koju razvijamo u Furijeov red po kosinusima grafički izgleda ovako



Neka je $f(x)$ 2π periodična f-ja.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$ Furijeov red f-je $f(x)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$ Furijeovi koeficijenti f-je $f(x)$

Ako je $f(x)$ parna tada je $f(x)\sin nx$ neparna $\Rightarrow b_n = 0 \ \forall n \in \mathbb{Z}$

Ako je $f(x)$ neparna tada je $f(x)\cos nx$ neparna $\Rightarrow a_n = 0 \ \forall n \in \mathbb{Z}$

Trebamo napraviti parno produženje f-je $f(x)$ (novu f-ju nazovimo $f^*(x)$)

$$f^*(x) = \begin{cases} f(x), & x \in (0,\pi) \\ f(-x), & x \in (-\pi, 0) \end{cases} = \begin{cases} \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2), & x \in (0,\pi) \\ \frac{1}{12}(3x^2 + 6\pi x + 2\pi), & x \in (-\pi, 0) \end{cases}$$

Izračunajmo Fourijeove koeficijente

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\,dx \overset{f^* \text{ parna}}{=\!=\!=} \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\,dx = \frac{1}{6\pi}\left(3\cdot\frac{1}{3}x^3\Big|_0^{\pi} - \right.$$

$$\left. - 6\pi \cdot \frac{1}{2}x^2\Big|_0^{\pi} + 2\pi^2 \cdot x\Big|_0^{\pi}\right) = \frac{1}{6\pi}(\pi^3 - 3\pi^3 + 2\pi^3) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\cos nx\,dx \overset{f^*(x) \text{ parna}}{=\!=\!=} \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\cos nx\,dx =$$

$$= \frac{1}{6\pi}\left(3\int_0^\pi x^2\cos nx\,dx - 6\pi\int_0^\pi x\cos nx\,dx + 2\pi^2\int_0^\pi \cos nx\,dx\right) \overset{(*)}{=}$$

$$I_1=\int_0^\pi x^2\cos nx\,dx = \left|\begin{matrix} u=x^2 & dv=\cos nx\,dx \\ du=2x\,dx & v=\frac{1}{n}\sin nx\end{matrix}\right| = \frac{1}{n}x^2\sin nx\Big|_0^\pi - \frac{2}{n}\int_0^\pi x\sin nx\,dx =$$

$$= \left|\begin{matrix} u=x & dv=\sin nx\,dx \\ du=dx & v=-\frac{1}{n}\cos nx\end{matrix}\right| = \frac{1}{n}\underbrace{(\pi^2\sin n\pi - 0)}_{=0} - \frac{2}{n}\left(-\frac{1}{n}x\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right)$$

$$= \frac{2}{n^2}(\pi\cos n\pi - 0) - \frac{2}{n^2}\sin nx\Big|_0^\pi = (-1)^n\frac{2\pi}{n^2}$$

$$I_2=\int_0^\pi x\cos nx\,dx = \left|\begin{matrix} u=x & dv=\cos nx\,dx \\ du=dx & v=\frac{1}{n}\sin nx\end{matrix}\right| = \frac{1}{n}x\sin nx\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx =$$

$$= \frac{1}{n}\underbrace{(\pi\sin n\pi - 0)}_{=0} - \frac{1}{n}\left(-\frac{1}{n}\right)\cos nx\Big|_0^\pi = \frac{1}{n^2}(\cos n\pi - \cos 0) = \frac{1}{n^2}((-1)^n-1)$$

$$I_3=\int_0^\pi \cos nx\,dx = \frac{1}{n}\sin nx\Big|_0^\pi = \frac{1}{n}(\sin n\pi - \sin 0) = 0$$

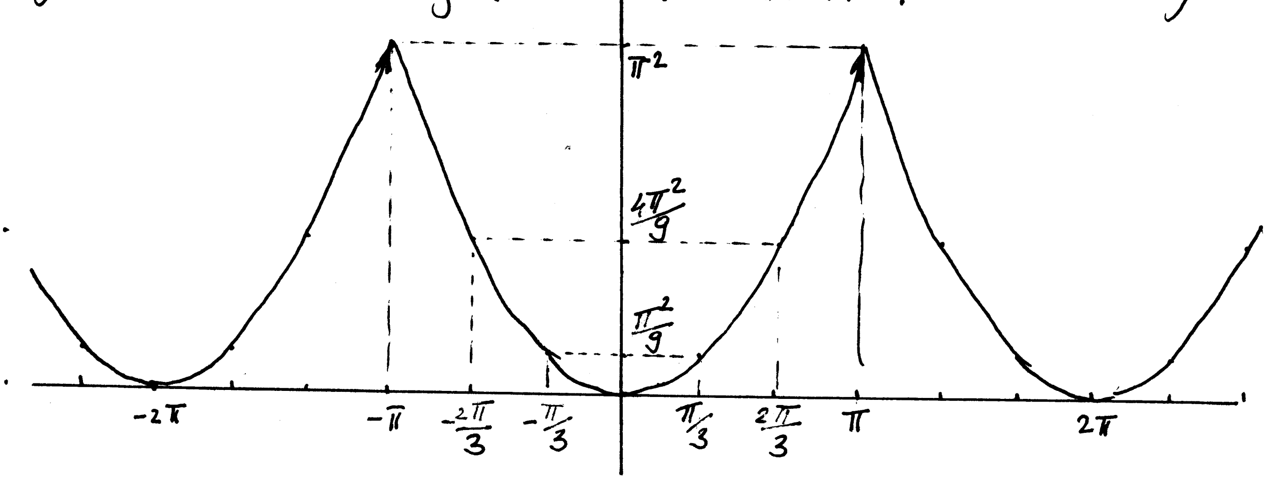
$$\overset{(*)}{=} \frac{1}{2\pi}(-1)^n\frac{2\pi}{n^2} - \frac{1}{n^2}((-1)^n-1) = \frac{1}{n^2}((-1)^n-(-1)^n+1) = \frac{1}{n^2}$$

$$f(x) = \sum_{n=1}^\infty \frac{1}{n^2}\cos nx, \quad x\in(0,\pi)$$

razvoj f-je f(x) u red po kosinusima

(Primjetimo da dobijeni rezultat možemo iskoristiti za sumiranje reda $\sum_{n=1}^\infty \frac{1}{n^2}$. Naime ako stavimo $x=0$ imamo $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$).

F-ju definisanu grafikom razviti u Furijer-ov red.



Dobijeni rezultat iskoristiti za sumiranje redova

a) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots$; b) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots$.

Rj. Primjetimo da je f-ja periodična perioda 2π, pa je možemo razviti u Furijeov red.

Posmatrajmo f-ju na intervalu $(-\pi,\pi)$.

$f(-\pi)=\pi^2$ $\quad f(0)=0$

$f(-\frac{2\pi}{3})=\frac{4\pi^2}{9}$ $\quad f(\frac{\pi}{3})=\frac{\pi^2}{}$

$f(-\frac{\pi}{3})=\frac{\pi^2}{9}$

Primjetimo da je $f(x)=x^2$, za $x\in(-\pi,\pi)$.

Dovoljno ju je pretvoriti u Furijer-ov red na ovom intervalu.

Prisjetimo se: Trigonometrijski red oblika

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right),\quad x\in[a,b]$$

zovemo Furijer-ov red f-je $f(x)$ na intervalu $[a,b]$, gdje su $a_0=\frac{2}{b-a}\int_a^b f(x)\,dx$, $a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx$

i $b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$ brojevi koje zovem Furijer-ovi koeficijenti.

Posmatramo interval $[-\pi,\pi]$, $b-a=2\pi$, $\frac{2}{b-a}=\frac{1}{\pi}$, $\frac{2n\pi x}{b-a}=nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{1}{\pi}\cdot\frac{1}{3}x^3\Big|_{-\pi}^{\pi}=\frac{2\pi^3}{3\pi}=\frac{2}{3}\pi^2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{parna}}{x^2}\ \underset{\text{parna}}{\cos nx}}_{\text{parna}}\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx=\begin{vmatrix}u=x^2 & dv=\cos nx\,dx\\ du=2x & v=\frac{1}{n}\sin nx\end{vmatrix}$$

$$=\frac{2}{\pi}\Big[\underbrace{\frac{1}{n}x^2\sin nx\Big|_0^{\pi}}_{=0-0}-\frac{2}{n}\int_0^{\pi}x\sin nx\,dx\Big]=\begin{vmatrix}u=x & dv=\sin nx\,dx\\ du=dx & v=\frac{-1}{n}\cos nx\end{vmatrix}=$$

$$=-\frac{4}{n\pi}\Big[-\frac{1}{n}x\cos nx\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx\Big]=\frac{4}{n^2\pi}(\pi\cos n\pi-0)+\underbrace{\frac{1}{n^2}\sin nx\Big|_0^{\pi}}_{=0-0}$$

$$=(-1)^n\frac{4}{n^2},\quad n\neq 0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{parna}}{x^2}\ \underset{\text{neparna}}{\sin nx}}_{=\text{neparna}}\,dx=0$$

traženi Furijer-ov red ↙

Prema tome $$x^2\sim\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}$$

Za $x=0$ imamo

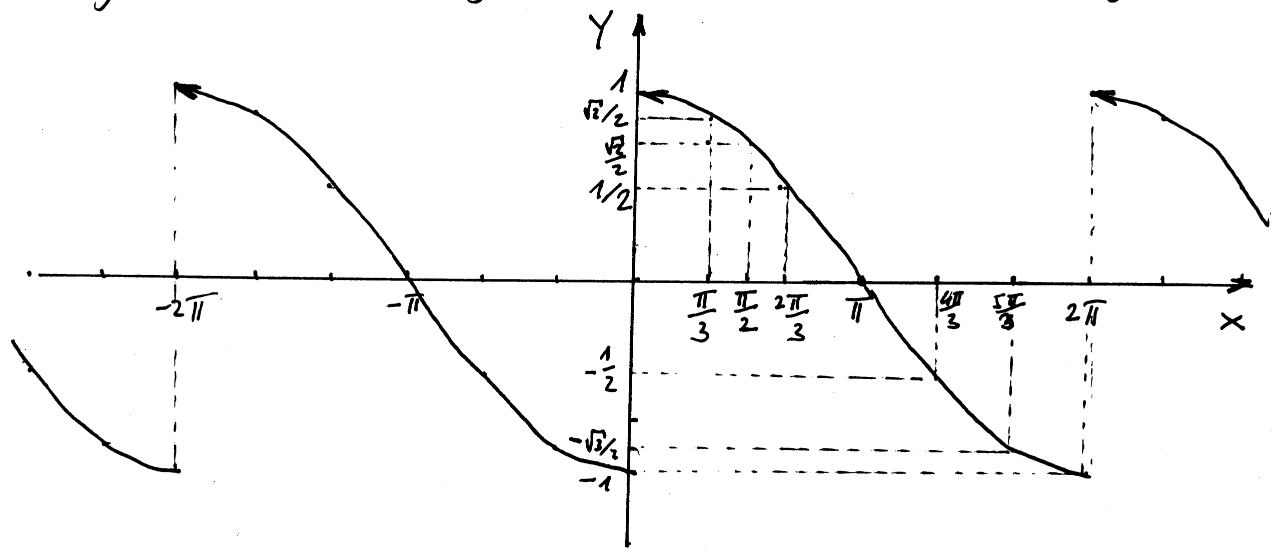
$$0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi^2}{3}+4\Big(\frac{-1}{1}+\frac{1}{2^2}-\frac{1}{3^2}+\dots\Big)$$

$$\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots=\frac{\pi^2}{12}$$

Za $x=\pi$ imamo

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots+\frac{1}{n^2}+\dots=\frac{\pi^2}{6}$$

(#) Funkciju definisanu grafikom razviti u Furijerov red



Dobijeni rezultat iskoristiti za sumiranje reda

$$\frac{1}{1\cdot 3}-\frac{3}{5\cdot 7}+\dots+\frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)}+\dots$$

Rj: Prikazana f-ja je periodična, perioda 2π pa je možemo pretvoriti u Furijer-ov red. Datu f-ju označimo sa $y=f(x)$. Primjetimo sa slike da imamo $f(\frac{\pi}{3})=\frac{\sqrt{3}}{2}$, $f(\frac{\pi}{2})=\frac{\sqrt{2}}{2}$, $f(\frac{2\pi}{3})=\frac{1}{2}$, $f(\pi)=0$.
Kako je $\cos(\frac{\pi}{6})=\frac{\sqrt{3}}{2}$, $\cos(\frac{\pi}{4})=\frac{\sqrt{2}}{2}$, $\cos(\frac{\pi}{3})=\frac{1}{2}$, $\cos(\frac{\pi}{2})=0$
to možemo primjetiti da je data f-ja $y=\cos(\frac{x}{2})$, $0\le x\le 2\pi$
Trigonometrijski red oblika $\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$
$x\in[a,b]$ nazivamo Furijer-ov red na intervalu $[a,b]$ gdje su

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\quad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx,$$

$n=1,2,\dots$ Furijer-ovi koeficijenti. U našem slučaju posmatramo interval $[0,2\pi]$ pa imamo $b-a=2\pi$, $\frac{2}{b-a}=\frac{1}{\pi}$, $\frac{2n\pi x}{b-a}=nx$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx,\quad b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \cos\frac{x}{2}\,dx = \frac{1}{\pi}\,2\int_0^{2\pi}\cos\frac{x}{2}\,d\left(\frac{x}{2}\right) = \frac{2}{\pi}\sin\frac{x}{2}\Big|_0^{2\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \left|\begin{array}{l}\text{data f-ja } f(x) \\ \text{je periodična}\end{array}\right| = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \left|\begin{array}{l}\text{prema datoj slici} \\ f(x) \text{ je neparna.} \\ \text{Kako je } \cos nx \\ \text{parna to je} \\ f(x)\cos nx \text{ neparna} \\ \text{f-ja, i imamo} \\ \text{simetričan interval}\end{array}\right|$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\cos\left(\frac{x}{2}\right)\sin nx\,dx = \left|\begin{array}{l}\sin(A+B) = \sin(A)\cos B + \sin B\cos A \\ \sin(A-B) = \sin A\cos B - \sin B\cos A \\ \hline 2\sin A\cos B = \sin(A+B) + \sin(A-B)\end{array}\right|$$

$$= \frac{1}{\pi}\cdot\frac{1}{2}\int_0^{2\pi}\left(\sin\left(nx+\frac{x}{2}\right) + \sin\left(nx-\frac{x}{2}\right)\right)dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\sin\left(n+\frac{1}{2}\right)x + \sin\left(n-\frac{1}{2}\right)x\right)dx$$

$$= \frac{1}{2\pi}\cdot\frac{(-1)}{n+\frac{1}{2}}\cos\left(n+\frac{1}{2}\right)x\Big|_0^{2\pi} + \frac{1}{2\pi}\cdot\frac{(-1)}{n-\frac{1}{2}}\cos\left(n-\frac{1}{2}\right)x\Big|_0^{2\pi} =$$

$$= \frac{(-1)}{\pi(2n+1)}\Big(\overbrace{\cos(2n+1)\pi - \cos 0}^{-2}\Big) + \frac{(-1)}{\pi(2n-1)}\Big(\overbrace{\cos(2n-1)\pi - \cos 0}^{-2}\Big) = \frac{2}{\pi(2n+1)} + \frac{2}{\pi(2n-1)}$$

$$= \frac{8n}{\pi(2n-1)(2n+1)}$$

Prema tome

$$\cos\frac{x}{2} \sim \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)}\sin nx$$ traženi Furijerov red

Ako za x uzmemo $\frac{\pi}{2}$ imamo

$$\frac{8}{\pi}\left(\frac{1}{1\cdot 3} + \frac{2\cdot 0}{3\cdot 5} + \frac{3\cdot(-1)}{5\cdot 7} + \ldots + \frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)} + \ldots\right) = \overset{\frac{\sqrt{2}}{2}}{\cos\frac{\pi}{4}}$$

Prema tome

$$\frac{1}{1\cdot 3} - \frac{3}{5\cdot 7} + \ldots + \frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)} + \ldots = \frac{\pi\sqrt{2}}{16}$$ tražena suma

(#) Neka je data f-ja $f: \mathbb{R}^2 \to \mathbb{R}$ definisana na sljedeći način

$$f(x,y) = \begin{cases} \dfrac{(xy)^2}{(xy)^2 + (x-y)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

Odrediti da li sljedeći limesi postoje i izračunati one limese koji postoje:

a) $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right]$; $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$;

b) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Rj.

a) $$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{(xy)^2}{(xy)^2+(x-y)^2}\right] =$$

$$= \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{(xy)^2}{(xy)^2+(x-y)^2}\right] = \lim_{y\to 0} \frac{0}{y^2} =$$

$$= \lim_{y\to 0} 0 = 0$$

b) Pokazaćemo da limes $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ ne postoji. Posmatrajmo dva niza tački $M_n\left(\frac{1}{n}, \frac{1}{n}\right)$ i $P_n\left(\frac{2}{n}, \frac{1}{n}\right)$, $n=1,2,\ldots$ koje teže $(0,0)$ kad $n\to\infty$.

$$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^4}}{\frac{1}{n^4} + \left(\frac{1}{n}-\frac{1}{n}\right)^2} = \lim_{n\to\infty} \frac{\frac{1}{n^4}}{\frac{1}{n^4}} = \lim_{n\to\infty} 1 = 1$$

$$\lim_{n\to\infty} f\left(\frac{2}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{1/n^4}{1/n^4 + \frac{1}{n^2}} \begin{matrix} /\cdot n^4 \\ /\cdot n^4 \end{matrix} = \lim_{n\to\infty} \frac{1}{1+n^2} = 0$$

Prema tome limes zavisi od načina približavanja ka tački $(0,0)$ pa ne postoji.

(#) Ispitati neprekidnost f-je $f(x,y)=\begin{cases}\frac{x^2y^3}{2x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Rj.
Jedina tačka u kojoj f-ja $f(x,y)$ može imati prekid je tačka $(0,0)$. F-ja će biti neprekidna u ovoj tački akko

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

tj. akko $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} = 0$.

Posmatrajmo približavanje tački $(0,0)$ preko prave $y=0$:

$$\lim_{(x,0)\to(0,0)} \frac{x^2\cdot 0}{2x^2+0^2} = \lim_{(x,0)\to(0,0)} \frac{0}{2x^2} = 0.$$

Posmatrajmo približavanje tački $(0,0)$ preko niza tački $\left(\frac{1}{n},\frac{1}{n}\right)$, $n\to\infty$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left|\begin{array}{l}\text{posmatramo}\\ \text{niz tački}\end{array}\right| = \lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot\frac{1}{n^3}}{\frac{2}{n^2}+\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\frac{1}{n^5}}{\frac{3}{n^2}} = \lim_{n\to\infty}\frac{n^2}{3n^5}\,\frac{/:n^2}{/:n^2} = 0.$$

Posmatrajmo približavanje tački $(0,0)$ preko prave $y=mx$:

$$\lim_{(x,mx)\to(0,0)} \frac{x^2\cdot m^3x^3}{2x^2+m^2x^2}\,\frac{/:x^2}{/:x^2} = \lim_{(x,mx)\to(0,0)} \frac{x^3m^3}{2+m^2} = 0.$$

Odavde možemo naslutiti da je vrijednost ovog limesa u tački $(0,0)$ jednaka 0.

ZAŠTO?

$$0 \le \left|\frac{x^2y^3}{2x^2+y^2}\right| = \frac{x^2|y^3|}{2x^2+y^2} \le |y^3| \to 0 \text{ kad } y\to 0$$

tj. kad $(x,y)\to(0,0)$

Prema teoremi "dva policajca" $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} = 0$.

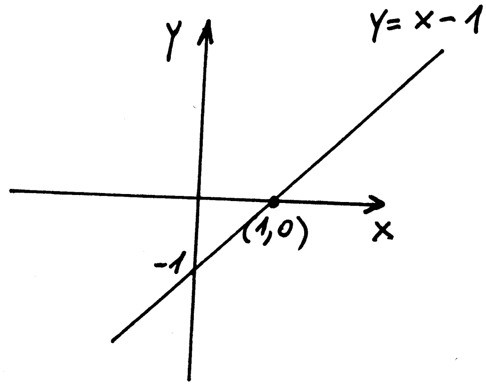
Data f-ja je neprekidna (u svakoj tački).

Ispitati neprekidnost f-je

$$f(x,y)=\begin{cases}\dfrac{(x-1)^2\ln x}{(x-1)^2+y^2}, & (x,y)\neq(1,0)\\[2mm] 0, & (x,y)=(1,0)\end{cases}.$$

R: Jedina sumnjiva tačka u kojoj f-ja može imati prekid je tačka $(1,0)$. F-ja će biti neprekidna u ovoj tački akko

$$\lim_{(x,y)\to(1,0)} f(x,y)=f(1,0)$$



tj. akko $\displaystyle\lim_{(x,y)\to(1,0)}\frac{(x-1)^2\ln x}{(x-1)^2+y^2}=0.$

Posmatrajmo približavanje tački $(1,0)$ preko prave $y=0$.

$$\lim_{(x,0)\to(1,0)}\frac{(x-1)^2\ln x}{(x-1)^2+0^2}=\lim_{(x,0)\to(1,0)}\ln x=0$$

Posmatrajmo približavanje tački $(1,0)$ preko prave $y=x-1$

$$\lim_{(x,x-1)\to(1,0)}\frac{(x-1)^2\ln x}{(x-1)^2+(x-1)^2}=\lim_{(x,x-1)\to(1,0)}\frac{\ln x}{2}=0$$

Odavde možemo naslutiti da je možda vrijednost ovog limesa u tački $(1,0)$ jednaka 0.

Prisjetimo se teoreme "dva policajca":

$$\forall(x,y)\in\mathbb{R}^2\quad g(x,y)\leq f(x,y)\leq h(x,y)\ \text{ i }\ \lim_{(x,y)\to(a,b)}g(x,y)=\lim_{(x,y)\to(a,b)}h(x,y)=M$$

$$\Rightarrow \lim_{(x,y)\to(a,b)}f(x,y)=M \iff |f(x,y)-M|\to 0,\ (x,y)\to(a,b)$$

$(x+1)^2\geq 0\ \ \forall x\in\mathbb{R}$

$(x-1)^2+y^2\geq 0\ \ \forall x,y\in\mathbb{R}$

$$(x-1)^2+y^2\geq(x+1)^2\Rightarrow 0\leq\frac{(x-1)^2}{(x-1)^2+y^2}\leq 1\Rightarrow$$

$$\Rightarrow 0\leq\frac{(x-1)|\ln x|}{(x-1)^2+y^2}\leq|\ln x|\to 0,\ (x,y)\to(1,0)\ \overset{\text{Teor. dva polic.}}{\Longrightarrow}\ \lim_{(x,y)\to(1,0)}\frac{(x-1)^2\ln x}{(x-1)^2+y^2}=0$$

F-ja je neprekidna.

(#) Ako je $u = \frac{\varphi(x-y) + \psi(x+y)}{x}$, gdje su φ i ψ diferencijabilne f-je izračunati $\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) - x^2 \frac{\partial^2 u}{\partial y^2}$.

Rj.

$$u = \frac{1}{x}\left(\varphi(x-y) + \psi(x+y)\right) = x^{-1}\left(\varphi(x-y) + \psi(x+y)\right)$$

$$u'_x = \frac{\partial u}{\partial x} = (-1)x^{-2}\left(\varphi(\underbrace{x-y}_{s}) + \psi(\underbrace{x+y}_{t})\right) + \frac{1}{x}\left(\varphi'_s \cdot s'_x + \psi'_t \cdot t'_x\right) =$$

$$= \frac{-1}{x^2}\left[\varphi(x-y) + \psi(x+y)\right] + \frac{1}{x}\left(\varphi'_s \cdot 1 + \psi'_t \cdot 1\right)$$

$$x^2 \frac{\partial u}{\partial x} = -\varphi(x-y) - \psi(x+y) + x\left(\varphi'_s + \psi'_t\right)$$ gdje su $s = x-y$ i $t = x+y$

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = \underline{-\varphi'_s \cdot 1} \underline{\underline{- \psi'_t \cdot 1}} \underline{+ 1 \cdot (\varphi'_s} \underline{\underline{+ \psi'_t}}) + x\left(\varphi''_{ss} \cdot 1 + \psi''_{tt} \cdot 1\right)$$

$$= x\left(\varphi''_{ss} + \psi''_{tt}\right) \qquad \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x}\left(\varphi'_s \cdot s'_y + \psi'_t \cdot t'_y\right) = \frac{1}{x}\left(\varphi'_s \cdot (-1) + \psi'_t \cdot 1\right) = \frac{1}{x}\left(-\varphi'_s + \psi'_t\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x}\left(-\varphi''_{ss} \cdot s'_y + \psi''_{tt} \cdot t'_y\right) = \frac{1}{x}\left(\varphi''_{ss} + \psi''_{tt}\right)$$

$$x^2 \frac{\partial^2 u}{\partial y^2} = x\left(\varphi''_{ss} + \psi''_{tt}\right) \qquad \ldots (2)$$

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) - x^2 \frac{\partial^2 u}{\partial y^2} \overset{(1) \text{ i } (2)}{=\!=} 0$$ traženo rješenje

(#) Ako je $z = \frac{y}{f(x^2-y^2)}$, gde je f diferencijabilna f-ja,

izračunati $\frac{1}{x} \cdot \frac{\partial z}{\partial x} + \frac{1}{y} \cdot \frac{\partial z}{\partial y}$

Rj. $z = y f^{-1}(x^2-y^2) = y f^{-1}(u)$, gde je $u = x^2-y^2$

$$\frac{\partial z}{\partial x} = y(-1) f_u^{-2}(x^2-y^2) \cdot 2x = \frac{-2xy}{f_u^2(x^2+y^2)}$$

$$\frac{\partial z}{\partial y} = \left(y f^{-1}(u)\right)'_y = 1 \cdot f^{-1}(u) + y \cdot (-1) f_u^{-2}(u) \cdot (-2y) =$$

$$= \frac{1}{f(x^2-y^2)} + \frac{2y^2}{f_u^2(x^2+y^2)}$$

$$\frac{1}{x} \cdot \frac{\partial z}{\partial x} + \frac{1}{y} \cdot \frac{\partial z}{\partial y} = \frac{-2y}{f_u^2(x^2+y^2)} + \frac{1}{y f(x^2-y^2)} + \frac{2y}{f_u^2(x^2+y^2)} =$$

$$= \frac{1}{y f(x^2-y^2)} = \frac{1}{y^2} \cdot \frac{y}{f(x^2-y^2)} = \frac{z}{y^2}$$

prema tome $\frac{1}{x} \cdot \frac{\partial z}{\partial x} + \frac{1}{y} \cdot \frac{\partial z}{\partial y} = \frac{z}{y^2}$.

(#) Ako je $z = e^{y}\varphi\left(y e^{\frac{x^2}{2y^2}}\right)$ gdje je φ diferencijabilna f-ja, dokazati da je $(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xyz$.

Rj.

$$z = e^{y}\varphi(\xi), \text{ gdje je } \xi(x,y) = y e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial x} = y e^{\frac{x^2}{2y^2}} \cdot 2 \cdot \frac{x}{2y^2} = \frac{x}{y} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial y} = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2 y^{-2}\right)'_y = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2\cdot(-2)y^{-3}\right)$$

$$= e^{\frac{x^2}{2y^2}} - \frac{x^2}{y^2} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial z}{\partial x} = e^{y}\cdot\frac{\partial \varphi}{\partial \xi}\cdot\frac{\partial \varphi}{\partial x} = \frac{x}{y} e^{y} e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$\frac{\partial z}{\partial y} = e^{y}\varphi(\xi) + e^{y}\cdot\frac{\partial \varphi}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} = e^{y}\varphi(\xi) + e^{y}e^{\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi} - e^{y}\cdot\frac{x^2}{y^2} e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = (x^2-y^2)\cdot\frac{x}{y} e^{y+\frac{x^2}{2y^2}}\frac{\partial \varphi}{\partial \xi} + xy\left(e^{y}\varphi(\xi) + e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi} - \frac{x^2}{y^2} e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi}\right)$$

$$= \frac{x^3}{y} e^{y+\frac{x^2}{2y^2}}\frac{\partial \varphi}{\partial \xi} - yx e^{y+\frac{x^2}{2y^2}}\frac{\partial \varphi}{\partial \xi} + xy e^{y}\varphi(\xi) + xy e^{y+\frac{x^2}{2y^2}}\frac{\partial \varphi}{\partial \xi} - \frac{x^3}{y} e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi} =$$

$$= xy e^{y}\varphi(\xi) = xy e^{y}\varphi\left(y e^{\frac{x^2}{2y^2}}\right) = xyz$$

(#) Provjeriti da li f-ja $z=\operatorname{arctg}\frac{x}{y}$, u kojoj je $x=u+v$, $y=u-v$, zadovoljava jednakost

$$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{u-v}{v^2+u^2}.$$

Rj. Primjetimo da je f-ja z složena f-ja dvije promjenjive

$z=z(u,v)$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \qquad \frac{x}{y}=xy^{-1}$$

$$\frac{\partial z}{\partial x}=\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}=\frac{y^2}{x^2+y^2}\cdot\frac{1}{y}=\frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y}=\frac{1}{1+\left(\frac{x}{y}\right)^2}(-1)xy^{-2}=\frac{-y^2}{x^2+y^2}\cdot\frac{x}{y^2}=\frac{-x}{x^2+y^2}$$

$$\frac{\partial x}{\partial u}=1 \qquad \frac{\partial y}{\partial u}=1$$

$$\frac{\partial z}{\partial u}=\frac{y}{x^2+y^2}\cdot 1+\frac{(-x)}{x^2+y^2}\cdot 1=\frac{y-x}{x^2+y^2}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{y}{x^2+y^2}\cdot 1+\frac{-x}{x^2+y^2}\cdot(-1)=\frac{y+x}{x^2+y^2}$$

$$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{y-x+y+x}{x^2+y^2}=\frac{2y}{x^2+y^2}=\frac{2(u-v)}{u^2+2uv+v^2+u^2-2uv+v^2}=$$

$$=\frac{2(u-v)}{2(u^2+v^2)}=\frac{u-v}{u^2+v^2}$$

vrijedi data jednakost

(#) Ako je $f(x) = \arcsin \frac{x}{y}$ gdje je $y = \sqrt{x^2+1}$

provjeriti da li je $\frac{df}{dx} = \frac{1}{x^2+1}$.

Rj.

Primjetimo da je $f(x)$ složena f-ja jedne promjenjive.
Koristimo formulu

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} = \frac{1}{y\sqrt{\frac{y^2-x^2}{y^2}}} = \frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{-x}{y^2} = \frac{-x}{y\sqrt{y^2-x^2}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

$$\frac{df}{dx} = \frac{1}{\sqrt{y^2-x^2}} + \frac{-x^2}{\underbrace{y}_{=\sqrt{x^2+1}}\sqrt{y^2-x^2}\sqrt{x^2+1}} = \frac{1}{\sqrt{y^2-x^2}} - \frac{x^2}{(x^2+1)\sqrt{y^2-x^2}}$$

$$= \frac{1}{\underbrace{\sqrt{y^2-x^2}}_{=1}}\left(1 - \frac{x^2}{x^2+1}\right) = \frac{x^2+1-x^2}{x^2+1} = \frac{1}{x^2+1}$$

$y^2-x^2 = x^2+1-x^2 = 1$

data jednakost je tačna

(#) Ako je $z = \ln(e^x + e^t)$ gdje je $x = t^3$

izračunati $\frac{\partial z}{\partial t}$ i $\frac{dz}{dt}$.

Rj:

$\frac{\partial z}{\partial t}$ je parcijalni izvod po t-u

$$\frac{\partial z}{\partial t} = \frac{1}{e^x + e^t} \cdot e^t = \frac{e^t}{e^x + e^t}$$

$\frac{dz}{dt}$ je izvod po t složene f-je z (čija je promjenljiva t)

Izvod ćemo naći po formuli

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial t}$$

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^x + e^t}, \quad \frac{dx}{dt} = 3t^2$$

$$\frac{dz}{dt} = \frac{3t^2 e^x}{e^x + e^t} + \frac{e^t}{e^x + e^t} = \frac{3t^2 e^x + e^t}{e^x + e^t}$$

Provjeriti da li f-ja $u = \sin x + F(\sin y - \sin x)$ u kojoj je F diferencijabilna f-ja zadovoljava jednakost

$$\frac{\partial u}{\partial y}\cos x + \frac{\partial u}{\partial x}\cos y = \cos x \cos y .$$

Rj: Označimo sa $v = \sin y - \sin x$. Imamo

$$u = \sin x + F(v)$$

u je složena f-ja promjenjivih x i y.

Koristimo formulu

$$\frac{\partial u}{\partial x} = \frac{\partial \sin x}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial \sin x}{\partial y} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y}$$

Imamo:

$$\frac{\partial u}{\partial x} = \cos x + F'_v\cdot(-\cos x) \Rightarrow \frac{\partial u}{\partial x}\cdot\cos y = \cos x\cos y - \cos x\cos y\, F'_v$$

$$\frac{\partial u}{\partial y} = 0 + F'_v\cdot\cos y \Rightarrow \frac{\partial u}{\partial y}\cdot\cos x = \cos x\cos y\, F'_v \quad +$$

$$\frac{\partial u}{\partial y}\cos x + \frac{\partial u}{\partial x}\cos y = \cos x\cos y$$

vrijedi data jednakost

(#) Provjeriti da li f-ja $z = \varphi(x^2+y^2)$, u kojoj je φ diferencijabilna f-ja, zadovoljava jednakost

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$

R: Primjetimo da je z složena f-ja dvije promjenjive. Ako su u označimo x^2+y^2 tj. $u = x^2+y^2$ imamo

$$z = \varphi(u)$$

$$\frac{\partial z}{\partial x} = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial \varphi}{\partial u} \cdot 2x = \varphi_u' \cdot 2x$$

$$\frac{\partial z}{\partial y} = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial y} = \frac{\partial \varphi}{\partial u} \cdot 2y = \varphi_u' \cdot 2y$$

$$y \cdot \frac{\partial z}{\partial x} = 2xy\,\varphi_u'$$

$$x \cdot \frac{\partial z}{\partial y} = 2xy\,\varphi_u'$$

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$ vrijedi data jednakost

(#) Ako je $p = u^2 \ln v$ pri čemu je $u = \frac{x}{y}$ i $v = 3x - 2y$, odrediti $\frac{\partial p}{\partial x}$ i provjeriti da li vrijedi $\frac{\partial p}{\partial y} = -\frac{2xu}{vy^2}(v \ln v + y)$.

Rj.

Vidimo da je p složena f-ja dvije promjenjive x i y. Koristićemo sljedeću formulu

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial p}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial p}{\partial v} \cdot \frac{\partial v}{\partial y}.$$

$$\frac{\partial p}{\partial u} = 2u \ln v, \quad \frac{\partial p}{\partial v} = \frac{u^2}{v}, \quad \frac{\partial v}{\partial x} = 3, \quad \frac{\partial v}{\partial y} = -2$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}, \quad \frac{\partial u}{\partial y} = x(-1)y^{-2} = -\frac{x}{y^2}$$

Prema tome

$$\left[\frac{3u^2}{v} = \frac{u}{vy} 3yu = \frac{u}{vy} 3x \right] \quad u = \frac{x}{y}$$

$$\frac{\partial p}{\partial x} = \frac{2u \ln v}{y} + \frac{3u^2}{v} = \frac{u}{vy}(2v \ln v + 3x)$$

$$\frac{\partial p}{\partial y} = \frac{-2u \ln v \, x}{y^2} - \frac{2u^2}{v} = -\frac{2xu}{vy^2}(v \ln v + y)$$

vrijedi data jednakost

$$\left[-\frac{2u^2}{v} = -\frac{2xu}{vy^2} \cdot \underbrace{y^2 \cdot u \cdot \frac{1}{x}}_{=y} \right]$$

(#) Razložiti f-ju $f(x,y)=\operatorname{arctg}(x^2y-2e^{x-1})$ po formuli Tejlova u okolini tačke $M(1,3)$ do stepena drugog reda zaključno.

Rj. Prisjetimo se kako izgleda Tejlorova formula za f-ju dvije promjenjive u okolini tačke $M_0(x_0,y_0)$

$$f(x,y)=f(x_0,y_0)+\sum_{k=1}^{n}\frac{1}{k!}\left((x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right)^k f(x_0,y_0)+R_n(x_0,y_0)$$

$$=f(x_0,y_0)+f'_x(x_0,y_0)(x-x_0)+f'_y(x_0,y_0)(y-y_0)+$$

1
1 1
1 2 1
1 3 3 1

$$+\frac{1}{2}\left(f''_{xx}(x_0,y_0)(x-x_0)^2+2f''_{xy}(x_0,y_0)(x-x_0)(y-y_0)+f''_{yy}(x_0,y_0)(y-y_0)^2\right)$$

$$+\frac{1}{6}\left(\frac{\partial^3 f(x_0,y_0)}{\partial x^3}(x-x_0)^3+3\frac{\partial^3 f(x_0,y_0)}{\partial x^2\partial y}(x-x_0)^2(y-y_0)+3\frac{\partial^3 f(x_0,y_0)}{\partial x\partial y^2}(x-x_0)(y-y_0)^2+\frac{\partial^3 f(x_0,y_0)}{\partial y^3}(y-y_0)^3\right)$$

$$+\dots$$

$$\frac{\partial f}{\partial x}=\frac{2xy-2e^{x-1}\cdot 1}{1+(x^2y-2e^{x-1})^2}=2\,\frac{xy-e^{x-1}}{1+(x^2y-2e^{x-1})^2}$$

$$\frac{\partial f}{\partial y}=\frac{x^2-0}{1+(x^2y-2e^{x-1})^2}=\frac{x^2}{1+(x^2y-2e^{x-1})^2}$$

$$\frac{\partial^2 f}{\partial x^2}=2\left(\frac{(y-e^{x-1})\cdot(1+(x^2y-2e^{x-1})^2)-(xy-e^{x-1})\cdot 2(x^2y-2e^{x-1})\cdot(2xy-2e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}\right)$$

$$=2\,\frac{y-e^{x-1}}{1+(x^2y-2e^{x-1})^2}-8\,\frac{(xy-e^{x-1})(x^2y-2e^{x-1})(xy-e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$=2\,\frac{y-e^{x-1}}{1+(x^2y-2e^{x-1})^2}-8\,\frac{(xy-e^{x-1})^2(x^2y-2e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$\frac{\partial^2 f}{\partial x\partial y}=2\,\frac{x(1+(x^2y-2e^{x-1})^2)-(xy-e^{x-1})\cdot 2(x^2y-2e^{x-1})x^2}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$=2\,\frac{x}{1+(x^2y-2e^{x-1})^2}-4\,\frac{x^2(xy-e^{x-1})(x^2y-2e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x^2\, 2(x^2y-2e^{x-1})\cdot x^2}{(1+(x^2y-2e^{x-1})^2)^2} = -2\,\frac{x^4(x^2y-2e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$\frac{\partial f}{\partial x}(1,3) = 2\cdot\frac{3-e^0}{2} = 2$$

$1+(x^2y-2e^{x-1})^2$ za $x=1,\ y=3$
$= 1+(3-2\cdot e^0)^2 = 1+1 = 2$

$$\frac{\partial f}{\partial y}(1,3) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2}(1,3) = 2\cdot\frac{2}{2} - 8\cdot\frac{4\cdot(3-2)}{4} = 2-8 = -6$$

$$\frac{\partial^2 f}{\partial x\partial y}(1,3) = 2\cdot\frac{1}{2} - 4\cdot\frac{1\cdot 2\cdot 1}{4} = 1-2 = -1$$

$$\frac{\partial^2 f}{\partial y^2} = -2\cdot\frac{1\cdot(3-2)}{4} = -\frac{1}{2}$$

$$f(1,3) = \operatorname{arc\,tg}(3-2) = \operatorname{arc\,tg} 1 = \frac{\pi}{4}$$

$$\operatorname{arc\,tg}(x^2y-2e^{x-1}) = \frac{\pi}{4} + 2(x-1) + \frac{1}{2}(y-3) -$$

$$-3(x-1)^2 - (x-1)(y-3) - \frac{1}{4}(y-3)^2 + \ldots$$

(#) F-ju $f(x,y) = \operatorname{arctg} \frac{x-y}{1+xy}$ razviti u Tejlorov red do članova 4. reda u okolini tačke $(0,0)$. Prikazati izgled opšteg člana.

Rj. F-ja $z = f(x,y)$ razložena po formuli Tejlora u okolini tačke (ρ_1, ρ_2):

$$f(x,y) = f(\rho_1,\rho_2) + \sum_{k=1}^{\infty} \frac{1}{k!}\left((x-\rho_1)\frac{\partial}{\partial x} + (y-\rho_2)\frac{\partial}{\partial y}\right)^k f(\rho_1,\rho_2) =$$

$$= f(\rho_1,\rho_2) + \frac{1}{1!}\left[\frac{\partial f(\rho_1,\rho_2)}{\partial x}(x-\rho_1) + \frac{\partial f(\rho_1,\rho_2)}{\partial y}(y-\rho_1)\right] + \frac{1}{2!}\left[\frac{\partial^2 f(\rho_1,\rho_2)}{\partial x^2}(x-\rho_1)^2 + \right.$$

$$\left. + \frac{\partial^2 f(\rho_1,\rho_2)}{\partial x \partial y}(x-\rho_1)(y-\rho_2) + \frac{\partial^2 f(\rho_1,\rho_2)}{\partial y^2}(y-\rho_2)^2\right] + \frac{1}{3!}\left[\frac{\partial^3 f(\rho_1,\rho_2)}{\partial x^3}(x-\rho_1)^3 + 3\frac{\partial^3 f(\rho_1,\rho_2)}{\partial x^2 \partial y}(x-\rho_1)^2(y-\rho_2)\right.$$

$$\left. + 3\frac{\partial^3 f(\rho_1,\rho_2)}{\partial x \partial y^2}(x-\rho_1)(y-\rho_2)^2 + \frac{\partial^3 f(\rho_1,\rho_2)}{\partial y^2}(y-\rho_2)^3\right] + \frac{1}{4!}\cdot\left[\frac{\partial^4 f(\rho_1,\rho_2)}{\partial x^4}(x-\rho_1)^4 + \ldots\right.$$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x-y}{1+xy}\right)^2}\cdot\frac{(1+xy)-(x-y)\cdot y}{(1+xy)^2} = \frac{(1+xy)^2}{(1+xy)^2+(x-y)^2}\cdot\frac{1+xy-xy+y^2}{(1+xy)^2} = \frac{1+y^2}{1+2xy+x^2y^2+x^2-2xy+y^2} =$$

$$= \frac{1+y^2}{1+x^2+y^2+x^2y^2} = \frac{1+y^2}{1+x^2+y^2(1+x^2)} = \frac{1+y^2}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2}$$

$$\frac{\partial f}{\partial x}(0,0) = 1, \quad \frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x-y}{1+xy}\right)^2}\cdot\frac{(-1)(1+xy)-(x-y)x}{(1+xy)^2} = \frac{(1+xy)^2}{(1+xy)^2+(x-y)^2}\cdot\frac{-1-xy-x^2+xy}{(1+xy)^2}$$

$$= \frac{(-1)(1+x^2)}{1+2xy+x^2y^2+x^2-2xy+y^2} = \frac{(-1)(1+x^2)}{(1+x^2)(1+y^2)} = \frac{-1}{1+y^2}, \quad \frac{\partial f}{\partial y}(0,0) = -1$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-2x}{(1+x^2)^2}, \quad \frac{\partial^2 f}{\partial x^2}(0,0) = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2y}{(1+y^2)^2}, \quad \frac{\partial^2 f}{\partial y^2}(0,0) = 0$$

$$\frac{\partial^3 f}{\partial x^3} = -2\,\frac{(1+x^2)^2 - x\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4} = -2\,\frac{1+x^2-4x^2}{(1+x^2)^3} = -2\,\frac{1-3x^2}{(1+x^2)^3}, \quad \frac{\partial^3 f}{\partial x^3}(0,0) = -2$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = 0, \quad \frac{\partial^3 f}{\partial x \partial y^2} = 0, \quad \frac{\partial^3 f}{\partial y^3} = 2\,\frac{(1+y^2)^2 - y\cdot 2(1+y^2)\cdot 2y}{(1+y^2)^4} = 2\,\frac{1+y^2-4y^2}{(1+y^2)^3} = 2\,\frac{1-3y^2}{(1+y^2)^3}$$

$$\frac{\partial^3 f}{\partial y^3}(0,0) = 2, \quad \frac{\partial^4 f}{\partial x^4} = (-2)\cdot\frac{-6x(1+x^2)^3 - (1-3x^2)\cdot 3(1+x^2)^2\cdot 2x}{(1+x^2)^6} = (-2)\frac{-6x-6x^3-6x+18x^3}{(1+x^2)^4}$$

$$= (-2)\frac{-12x+12x^3}{(1+x^2)^4} = (-2)(-12)\frac{x-x^3}{(1+x^2)^4} = -24\,\frac{x(x^2-1)}{(x^2+1)^4}, \quad \frac{\partial^4 f}{\partial x^4}(0,0) = 0, \quad \frac{\partial^4 f}{\partial x^3 \partial y} = 0$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = 0, \quad \frac{\partial^4 f}{\partial x \partial y^3} = 0, \quad \frac{\partial^4 f}{\partial y^4} = 2\,\frac{-6y(1+y^2)^3 - (1-3y^2)3\cdot(1+y^2)^2\cdot 2y}{(1+y^2)} =$$

$$= 2\,\frac{-6y-6y^3-6y+18y^3}{(1+y^2)^4} = 2\,\frac{-12y+12y^3}{(1+y^2)^4} = 2\cdot 12\,\frac{-y+y^3}{(1+y^2)^4} = 24\,\frac{y(y^2-1)}{(y^2+1)^4}$$

$$\frac{\partial^4 f}{\partial y^4}(0,0) = 0, \quad f(0,0) = \operatorname{arctg} 0 = 0$$

$$f(x,y) = \frac{1}{1!}(x-y) + \frac{1}{2!}(0+0+0) + \frac{1}{3!}\left((-2)x^3 + 2y^3\right) + \frac{1}{4!}\cdot 0 + \dots$$

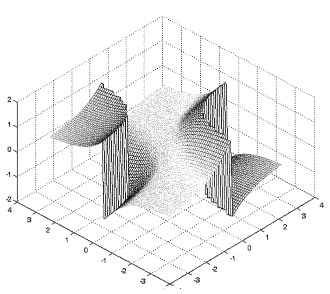
$$= x - y + \frac{-1}{3}(x^3-y^3) + \dots = x - y + \frac{(-1)^1}{3}(x^3-y^3) + \frac{(-1)^2}{5}(x^5-y^5)$$

$$+ \dots + \frac{(-1)^n}{2n+1}(x^{2n+1}-y^{2n+1}) + \dots$$

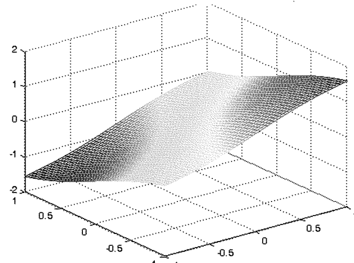
f-ja $f(x,y)$ razložena po formuli Tejlora

Dodatak.

Grafički prikažimo f-ju $f(x,y) = \operatorname{arctg}\frac{x-y}{1+xy}$ na intervalu



na intervalu [-pi,pi]×[-pi,pi]



na intervalu (-1,1)×(-1,1)

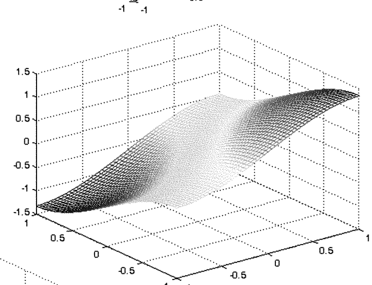
Grafički prikažimo sljedeće polinome:

$$f(x,y) = \sum_{n=0}^{1} \frac{(-1)^n}{2n+1}(x^{2n+1}-y^{2n+1})$$

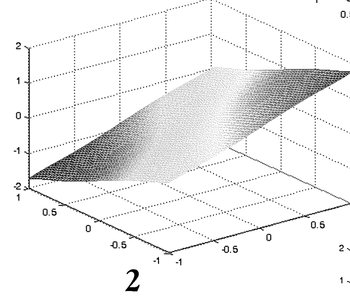
$$f(x,y) = \sum_{n=0}^{2} \frac{(-1)^n}{2n+1}(x^{2n+1}-y^{2n+1})$$

$$f(x,y) = \sum_{n=0}^{4} \frac{(-1)^n}{2n+1}(x^{2n+1}-y^{2n+1})$$

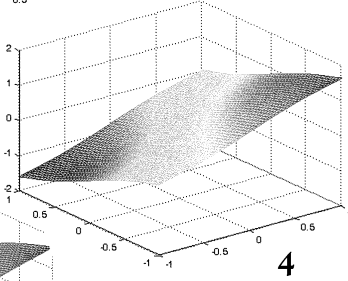
$$f(x,y) = \sum_{n=0}^{10} \frac{(-1)^n}{2n+1}(x^{2n+1}-y^{2n+1})$$



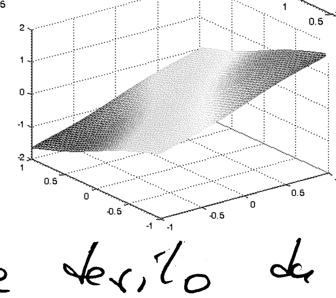
1



2



4



3

Šta možemo primjetiti. Šta bi se desilo da smo uzeli interval $[-\pi,\pi]\times[-\pi,\pi]$ (Kako bi izgledao graf?).

Odrediti jednačinu tangentne ravni na površ $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, koja je normalna na pravu $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.

Rj. Kako izgleda jednačina tangentne ravni i normale na neku površ $F(x,y,z)=0$?

$$F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

jednačina tangentne ravni na površ u tački $M(p_1,p_2,p_3)$

$$\frac{x-p_1}{F'_x(p_1,p_2,p_3)}=\frac{y-p_2}{F'_y(p_1,p_2,p_3)}=\frac{z-p_3}{F'_z(p_1,p_2,p_3)}$$

jednačina normale na površ $F(x,y,z)=0$ u tački $M(p_1,p_2,p_3)$

U našem slučaju $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$, a tačka u kojoj trebamo postaviti datu ravan nam nije poznata. Za trenutak, označimo tu tačku sa $M(x_0,y_0,z_0)$.

$$F'_x=\frac{2x}{a^2},\quad F'_y=\frac{2y}{b^2},\quad F'_z=\frac{2z}{c^2}$$

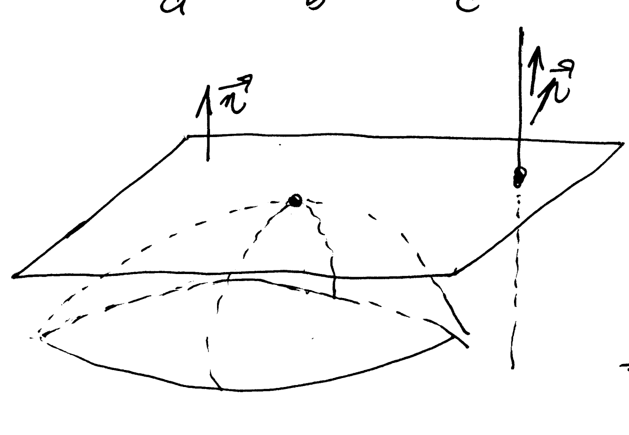
$$F'_x(M)(x-x_0)+F'_y(M)(y-y_0)+F'_z(M)(z-z_0)=0$$

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0 \quad /:2$$

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z-\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}\right)=0$$

Kako je tačka $M(x_0,y_0,z_0)$ tačka na elipsi imamo $\frac{x_0^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

tj. $\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z-1=0$.



Vektor normale tražene ravni je $\vec{n}\left(\frac{x_0}{a^2},\frac{y_0}{b^2},\frac{z_0}{c^2}\right)$.

$\vec{n}\parallel\vec{p}$ gdje je $\vec{p}=(1,2,3) \Rightarrow \vec{n}=k\cdot\vec{p}$

$\Rightarrow \vec{n}=(k,2k,3k)$, $k\in R$ tj. imamo

$$\frac{x_0}{a^2}=k, \quad \frac{y_0}{b^2}=2k, \quad \frac{z_0}{c^2}=3k \quad \Rightarrow \quad x_0=a^2k, \ y_0=2kb^2, \ z_0=3kc^2$$

Postavljamo još pitanje kako izračunati k?
M je tačka sa naše površi (sa elipse) pa

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1 \quad \text{tj.} \quad a^2k^2+4b^2k^2+9c^2k^2=1$$

$$k^2=\frac{1}{a^2+4b^2+9c^2}$$

$$k=\frac{\pm 1}{\sqrt{a^2+4b^2+9c^2}}$$

Na kraju imamo

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z-1=0$$

$$kx+2ky+3kz-1=0 \quad /:k$$

$$x+2y+3z-\frac{1}{k}=0$$

$$x+2y+3z\mp\sqrt{a^2+4b^2+9c^2}=0$$ jednačina tražene tangentne ravni

(#) Naći jednačinu tangentne ravni elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ koja na koordinatnim osama odsjeca jednake pozitivne odsječke.

Rj: Jednačina tangentne ravni na površ $F(x,y,z)=0$ u tački $M(p_1,p_2,p_3)$ ima jednačinu $F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)$

Nađimo jednačinu tangentne ravni na elipsoid u proizvoljnoj tački $M(p_1,p_2,p_3)$: (U našem slučaju $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$)

$$F'_x=\frac{1}{a^2}\cdot 2x=\frac{2x}{a^2},\quad F'_y=\frac{2y}{b^2},\quad F'_z=\frac{2z}{c^2}$$

$$F'_x(M)=\frac{2p_1}{a^2},\quad F'_y(M)=\frac{2p_2}{b^2},\quad F'_z(M)=\frac{2p_3}{c^2}$$

$$\frac{2p_1}{a^2}(x-p_1)+\frac{2p_2}{b^2}(y-p_2)+\frac{2p_3}{c^2}(z-p_3)=0 \quad /\cdot\frac{1}{2}$$

$$\frac{p_1}{a^2}x+\frac{p_2}{b^2}y+\frac{p_3}{c^2}z=\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}$$

Napišimo jednačinu ravni u kanonskom obliku

$$\frac{x}{\frac{a^2}{p_1}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{y}{\frac{b^2}{p_2}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{z}{\frac{c^2}{p_3}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}=1$$

Odavde možemo primjetiti da ako želimo da jednačina tangentne ravni na kordanatnim osama odsjeca jednake odsječke, potrebno i dovoljno je da $\frac{a^2}{p_1}=\frac{b^2}{p_2}$, $\frac{a^2}{p_1}=\frac{c^2}{p_3}$ i $\frac{b^2}{p_2}=\frac{c^2}{p_3}$. ...(*)

Isto tako primjetimo da je $\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}=1$ ...(**) (ZAŠTO?)

$(*)\Rightarrow p_1=\frac{a^2}{b^2}p_2$, $p_3=\frac{c^2}{b^2}p_2$ ...(1) Sad imamo:

$$\frac{x}{\frac{a^2}{\frac{a^2}{b^2}p_2}}+\frac{y}{\frac{b^2}{p_2}}+\frac{z}{\frac{c^2}{\frac{c^2}{b^2}p_2}}=1 \quad /:p_2$$

$$\frac{x}{b^2}+\frac{y}{b^2}+\frac{z}{b^2}=\frac{1}{p_2}$$

Kada (1) stavimo u (**) dobićemo da je $p_2=\frac{b^2}{\sqrt{a^2+b^2+c^2}}$

Prema tome:

$x+y+z=\sqrt{a^2+b^2+c^2}$ je jednačina tražene tangente

(#) Dokazati da tangentne ravni površi $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$ $(a>0)$ odsjecaju od koordinatnih osa odsječke čiji je zbir jednak a.

Rj. Jednačina tangentne ravni na površ $F(x,y,z)=0$ u tački $M(p_1,p_2,p_3)$ ima jednačinu

$$F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

Primjetimo da ako je data kriva u ravni $F(x,y)=0$ tada jednačina tangente u tački $N(c_1,c_2)$ ima jenačinu

$$F'_x(c_1,c_2)(x-c_1)+F'_y(c_1,c_2)(y-c_2)=0$$

npr. jednačina tangente na krivu $y=x^2+x-6$ u tački $(-3,0)$ je $y=-5x-15$.

U našem slučaju data površ je $F(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}-\sqrt{a}$

$$F'_x=\frac{1}{2\sqrt{x}},\quad F'_y=\frac{1}{2\sqrt{y}},\quad F'_z=\frac{1}{2\sqrt{z}}$$

Uzmimo proizvoljnu tačku sa površi $M(p_1,p_2,p_3)$.

$$F'_x(p_1,p_2,p_3)=\frac{1}{2\sqrt{p_1}}=\frac{\sqrt{p_1}}{2p_1};\quad F'_y(p_1,p_2,p_3)=\frac{1}{2\sqrt{p_2}}=\frac{\sqrt{p_2}}{2p_2};\quad F'_z(p_1,p_2,p_3)=\frac{1}{2\sqrt{p_3}}=\frac{\sqrt{p_3}}{2p_3}$$

Jednačina tangentne ravni na površ u tački M

$$\frac{\sqrt{p_1}}{2p_1}(x-p_1)+\frac{\sqrt{p_2}}{2p_2}(y-p_2)+\frac{\sqrt{p_3}}{2p_3}(z-p_3)=0$$

Napišimo jednačinu u kanonskom obliku

$$\frac{\sqrt{p_1}}{2p_1}x+\frac{\sqrt{p_2}}{2p_2}y+\frac{\sqrt{p_3}}{2p_3}z=\underbrace{\frac{1}{2}\sqrt{p_1}+\frac{1}{2}\sqrt{p_2}+\frac{1}{2}\sqrt{p_3}}_{\frac{1}{2}\sqrt{a}}$$

Primjetimo da je $\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}=\sqrt{a}$. ZAŠTO?

$/\cdot 2/\sqrt{a}$

$$\frac{x}{\frac{p_1\sqrt{a}}{\sqrt{p_1}}}+\frac{y}{\frac{p_2\sqrt{a}}{\sqrt{p_2}}}+\frac{z}{\frac{p_3\sqrt{a}}{\sqrt{p_3}}}=1$$

$$\frac{x}{\sqrt{p_1 a}}+\frac{y}{\sqrt{p_2 a}}+\frac{z}{\sqrt{p_3 a}}=1$$

Jednačina tangentne ravni na x-osi odsjeca $\sqrt{a}\cdot\sqrt{p_1}$, na y-osi $\sqrt{a}\cdot\sqrt{p_2}$ i na z-osi $\sqrt{a}\cdot\sqrt{p_3}$.

Zbir ovih odsječaka iznosi

$$\sqrt{a}\sqrt{p_1}+\sqrt{a}\sqrt{p_2}+\sqrt{a}\sqrt{p_3}=\sqrt{a}(\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3})=a$$

q.e.d.

(#) Napisati jednačinu tangentne ravni i normale na površ $2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$ u tački $M(2,2,1)$.

Rj: Ako površ S ima jednačinu u implicitnom obliku $F(x,y,z)=0$ tada jednačina tangentne ravni i normale na površ S u tački $M(p_1,p_2,p_3)$ se računaju po formuli:

$$\alpha: F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

$$n: \frac{x-p_1}{F'_x(p_1,p_2,p_3)} = \frac{y-p_2}{F'_y(p_1,p_2,p_3)} = \frac{z-p_3}{F'_z(p_1,p_2,p_3)}$$

$$2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$$

$$\left(\frac{x}{z}\right)'_z = (xz^{-1})'_z = (-1)xz^{-2}$$

$$F(x,y,z) = 2^{\frac{x}{z}} + 2^{\frac{y}{z}} - 8 = 0$$

$$F'_x = 2^{\frac{x}{z}} \ln 2 \cdot \frac{1}{z} \quad \Rightarrow F'_x(2,2,1) = 4\ln 2.$$

$$F'_y = 2^{\frac{y}{z}} \ln 2 \cdot \frac{1}{z} \quad \Rightarrow F'_y(2,2,1) = 4\ln 2$$

$$F'_z = 2^{\frac{x}{z}} \ln 2 \cdot \left(\frac{x}{z}\right)'_z + 2^{\frac{y}{z}} \ln 2 \left(\frac{y}{z}\right)'_z = -\frac{x}{z^2} 2^{\frac{x}{z}} \ln 2 - \frac{y}{z^2} 2^{\frac{y}{z}} \ln 2$$

$$= -\frac{1}{z^2} \ln 2 \left(x 2^{\frac{x}{z}} + y 2^{\frac{y}{z}}\right)$$

$$F'_z(2,2,1) = -\ln 2 (2 \cdot 4 + 2 \cdot 4) = -16 \ln 2$$

$$4\ln 2 (x-2) + 4\ln 2 (y-2) + (-16\ln 2)(z-1) = 0$$

$$4x\ln 2 + 4y\ln 2 - 16z\ln 2 + 8\ln 2 = 0$$ jednačina tangentne ravni

$$\frac{x-2}{4\ln 2} = \frac{y-2}{4\ln 2} = \frac{z-1}{-16\ln 2} \Rightarrow \frac{x-2}{1} = \frac{y-2}{1} = \frac{z-1}{-4}$$

jednačina normale na površ

(#) Izračunati izvod f-je $u = x^2y^2 + z^2 - 3xyz$ u tački $T(1,1,2)$ u smjeru koji čini s koordinatnim osama uglove $\frac{\pi}{3}$, $\frac{\pi}{4}$ i $\frac{\pi}{6}$.

Rj. Izvod f-je $u = f(x,y,z)$ u tački $M(p_1, p_2, p_3)$ u pravcu vektora $\vec{e}$ ($\vec{e}$ je jedinični vektor) se računa po formuli:

$$\frac{\partial u(M)}{\partial \vec{e}} = \operatorname{grad} u(M) \cdot \vec{e}$$

$$\frac{\partial u}{\partial x} = 2xy^2 - 3yz$$

$$\frac{\partial u}{\partial y} = 2x^2y - 3xz$$

$$\frac{\partial u}{\partial z} = 2z - 3xy$$

$$\operatorname{grad} u(M) = \left(\frac{\partial u}{\partial x}(M), \frac{\partial u}{\partial y}(M), \frac{\partial u}{\partial z}(M)\right)$$

$$\frac{\partial u}{\partial x}(1,1,2) = 2\cdot 1\cdot 1 - 3\cdot 1\cdot 2 = -4$$

$$\frac{\partial u}{\partial y}(1,1,2) = 2\cdot 1\cdot 1 - 3\cdot 1\cdot 2 = -4$$

$$\frac{\partial u}{\partial z}(1,1,2) = 2\cdot 2 - 3\cdot 1\cdot 1 = 1$$

$$\operatorname{grad} u(M) = (-4, -4, 1)$$

$$\vec{e} = \cos\alpha\, \vec{i} + \cos\beta\, \vec{j} + \cos\gamma\, \vec{k} = (\cos\alpha, \cos\beta, \cos\gamma)$$

$$\vec{e} = \left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right) \qquad \alpha = \frac{\pi}{3},\ \beta = \frac{\pi}{4},\ \gamma = \frac{\pi}{6}$$

$$\frac{\partial u}{\partial \vec{e}}(M) = (-4, -4, 1)\left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right) = -2 - 2\sqrt{2} + \frac{\sqrt{3}}{2}$$

$$|\vec{e}| = \sqrt{\frac{1}{4} + \frac{2}{4} + \frac{3}{4}} = 1$$

$$\frac{\partial u}{\partial \vec{e}}(M) = \frac{\sqrt{3}}{2} - 2\sqrt{2} - 2$$

traženo rješenje
(izvod f-je u u tački T u datom smjeru)

(#) Odrediti ekstreme f-je

$$f(x,y) = x^2 - xy + y^2 - 2x - 2y.$$

Rj. Odredimo parcijalne izvode

$$\frac{\partial f}{\partial x} = 2x - y - 2$$

$$\frac{\partial f}{\partial y} = -x + 2y - 2$$

Zatim da bi odredili stacionarne tačke riješimo sistem

$$2x - y - 2 = 0$$
$$-x + 2y - 2 = 0 \quad / \cdot 2$$

$$2x - y - 2 = 0$$
$$+ \quad -2x + 4y - 4 = 0$$

$$3y - 6 = 0$$
$$y = 2$$

$y = 2 \Rightarrow 2x - 2 - 2 = 0$

$x = 2$

Stacionarna tačka je $M(2,2)$.

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

Za $M(2,2)$ imamo

$A = 2, \; B = -1, \; C = 2$

$$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = 4 - 1 = 3 > 0$$

f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$$f_{min}(2,2) = 4 - 4 + 4 - 4 - 4 = -4$$

(#) Naći ekstreme f-je $z = x + y + 4 + 4\sin x \sin y$.

Rj.

$$\frac{\partial z}{\partial x} = 1 + 4\cdot\cos x \sin y$$

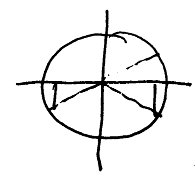
$$\frac{\partial z}{\partial y} = 1 + 4\sin x \cos y$$

$$1 + 4\cos x \sin y = 0$$
$$1 + 4\sin x \cos y = 0$$

$$\sin y \cos x = -\frac{1}{4} \quad (a)$$
$$\sin x \cos y = -\frac{1}{4} \quad (b)$$

(a)+(b): $\sin y \cos x + \sin x \cos y = -\frac{1}{2}$

$$\sin(y+x) = -\frac{1}{2}$$



$$y + x = \frac{7\pi}{6}$$
ili
$$y + x = \frac{11\pi}{6}$$

(a)-(b):

$$\sin y \cos x - \sin x \cos y = 0$$
$$\sin(y-x) = 0$$
$$y - x = 0 \quad \text{ili} \quad y - x = \pi$$

1° $x + y = \frac{7\pi}{6}$

$+\quad -x + y = 0$

$$2y = \frac{7\pi}{6}$$
$$y = \frac{7\pi}{12} \Rightarrow x = \frac{7\pi}{12}$$

2° $x + y = \frac{7\pi}{6}$

$-x + y = \pi$

$$2y = \frac{13\pi}{6}$$
$$y = \frac{13\pi}{12} \Rightarrow x = \frac{\pi}{12}.$$

3° $x + y = \frac{11\pi}{6}$

$-x + y = 0$

$$y = \frac{11\pi}{12} \Rightarrow$$
$$x = \frac{11\pi}{12}$$

4° $y + x = \frac{11\pi}{6}$

$+\quad y - x = \pi$

$$y = \frac{17\pi}{12} \Rightarrow x = \frac{5\pi}{12}$$

Stacionarne tačke su

$M_1\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right)$, $M_2\left(\frac{\pi}{12}, \frac{13\pi}{12}\right)$,

$M_3\left(\frac{11\pi}{12}, \frac{11\pi}{12}\right)$, $M_4\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right)$

$$\frac{\partial^2 z}{\partial x^2} = -4\sin x \sin y$$

$$\frac{\partial^2 z}{\partial x \partial y} = 4\cos x \cos y$$

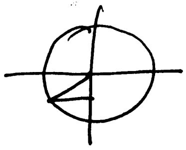
$$\frac{\partial^2 z}{\partial y^2} = -4\sin x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad (I)$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y \quad (II)$$

(I) + (II): $\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$

(II) − (I): $\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

• Za $M_1\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right)$



$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$A = -4\cdot\frac{1}{2}\left(\cos 0 - \cos\frac{7\pi}{6}\right) = -2\left(1+\frac{\sqrt{3}}{2}\right) = -2-\sqrt{3}$

$B = 4\cdot\frac{1}{2}\left(\cos\frac{7\pi}{6} + \cos 0\right) = 2\left(-\frac{\sqrt{3}}{2}+1\right) = -\sqrt{3}+2$

$C = -2-\sqrt{3}$

$D = AC - B^2 = (2+\sqrt{3})^2 - (2-\sqrt{3})^2 > 0 \Rightarrow$ f-ja ima ekstrem

$A < 0$ f-ja u tački M_1 ima maksimum.

$Z_{max}\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right) = \frac{7\pi}{12} + \frac{7\pi}{12} + 4 + 2 + \sqrt{3} = 6 + \sqrt{3} + \frac{7\pi}{6}$ traženi ekstrem

• Za $M_2\left(\frac{\pi}{12}, \frac{13\pi}{12}\right)$

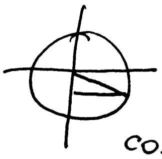
$A = -4\cdot\frac{1}{2}\left(\cos(-\pi) - \cos\frac{7\pi}{6}\right) = -2\left(-1+\frac{\sqrt{3}}{2}\right) = 2-\sqrt{3}$

$B = 4\cdot\frac{1}{2}\left(\cos\frac{7\pi}{6} + \cos(-\pi)\right) = 2\left(-\frac{\sqrt{3}}{2}-1\right) = -\sqrt{3}-2$

$C = 2-\sqrt{3}$

$D = AC - B^2 = (2-\sqrt{3})^2 - (2+\sqrt{3})^2 > 0 \Rightarrow$ f-ja u tački M_2 nema ekstrema

• Za $M_3\left(\frac{11\pi}{12}, \frac{11\pi}{12}\right)$



$\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

$A = -4\cdot\frac{1}{2}\left(\cos 0 - \cos\frac{11\pi}{6}\right) = -2\left(1-\frac{\sqrt{3}}{2}\right) = -2+\sqrt{3}$

$B = 4\cdot\frac{1}{2}\left(\cos\frac{11\pi}{6} + \cos 0\right) = 2\left(\frac{\sqrt{3}}{2}+1\right) = \sqrt{3}+2$

$C = \sqrt{3}-2$

$D = (\sqrt{3}-2)^2 - (\sqrt{3}+2)^2 < 0 \Rightarrow$ f-ja u tački M_3 nema ekstrem

• Za $M_4\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right)$

$A = -4\cdot\frac{1}{2}\left(\cos(-\pi) - \cos\frac{11\pi}{6}\right) = -2\left(-1-\frac{\sqrt{3}}{2}\right) = 2+\sqrt{3}$

$B = 4\cdot\frac{1}{2}\left(\cos\frac{11\pi}{6} + \cos(-\pi)\right) = 2\left(\frac{\sqrt{3}}{2}-1\right) = \sqrt{3}-2$

$C = 2+\sqrt{3}$, $D = AC - B^2 = (2+\sqrt{3})^2 - (\sqrt{3}-2)^2 > 0 \Rightarrow$ f-ja u tački M_4 ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$Z_{min}\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right) = \frac{5\pi}{12} + \frac{17\pi}{12} + 4 + (-2-\sqrt{3}) = 2-\sqrt{3} + \frac{11\pi}{6}$ traženi ekstrem

Naći ekstreme f-je $z=(2x^2+3y^2)e^{-(x^2+y^2)}$.

Rj.

$$\frac{\partial z}{\partial x} = 4x\cdot e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2x)=(4x-4x^3-6xy^2)e^{-x^2-y^2}$$

$$\frac{\partial z}{\partial y} = 6y e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2y)=(6y-4x^2y-6y^3)e^{-x^2-y^2}$$

$$2x(2-2x^2-3y^2)e^{-x^2-y^2}=0$$
$$2y(3-2x^2-3y^2)e^{-x^2-y^2}=0$$

$e^{-x^2-y^2}\neq 0 \ \forall (x,y\in R)$

$x=0$ i $y=0$, $M_1(0,0)$

ili

$x=0$ i $3-2x^2-3y^2=0$

$3y^2=3$, $y^2=1$, $y_{1,2}=\pm 1$

$M_2(0,-1)$

$M_3(0,1)$

ili

$2-2x^2-3y^2=0$ i $y=0$

$2x^2=2$, $x^2=1$, $x_{1,2}=\pm 1$

$M_4(-1,0)$

$M_5(1,0)$

ili

$$2-2x^2-3y^2=0$$
$$-\quad 3-2x^2-3y^2=0$$
$$-1=0$$

sistem nema rješenja

Stacionarne tačke su M_1, M_2, M_3, M_4 i M_5.

$$\frac{\partial^2 z}{\partial x^2}=(4-12x^2-6y^2)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2x)=(8x^4+12x^2y^2-20x^2-6y^2+4)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=(-12xy)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2y)=(-20xy+8x^3y+12xy^3)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial y^2}=(6-4x^2-18y^2)e^{-x^2-y^2}+(6y-4x^2y-6y^3)e^{-x^2-y^2}\cdot(-2y)=(-30y^2+12y^4+8x^2y^2-4x^2+6)e^{-x^2-y^2}$$

za $M_1(0,0)$, $A=4$, $B=0$, $C=6$, $D=AC-B^2=24>0$ ima ekstrem

$A>0$ ima minimum, $z_{min}(0,0)=0$

za $M_2(0,-1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum, $z_{max}(0,-1)=3e^{-1}$

za $M_3(0,1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum $z_{max}(0,1)=3e^{-1}$

za $M_4(-1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$, $D=AC-B^2=-16e^{-2}<0$

f-ja u tački $M_4(-1,0)$ nema ekstrem

za $M_5(1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$,

f-ja u tački $M_5(1,0)$ nema ekstrem

(#) Odrediti ekstreme f-je $f(x,y) = x e^{y + x\sin y}$.

Rj: Odredimo parcijalne izvode

$$\frac{\partial f}{\partial x} = e^{y+x\sin y} + x e^{y+x\sin y} \cdot \sin y = e^{y+x\sin y}(1 + x\sin y)$$

$$\frac{\partial f}{\partial y} = x e^{y+x\sin y} \cdot (1 + x\cos y)$$

Da bi odredili stacionarne tačke trebamo riješiti sljedeći sistem

$$e^{y+x\sin y}(1+x\sin y) = 0$$
$$x e^{y+x\sin y}(1+x\cos y) = 0$$

primjetimo da je $e^{y+x\sin y} > 0 \ \forall x, y$
$/ e^{y+x\sin y}$ obe jednačine

$$1 + x\sin y = 0$$
$$x(1 + x\cos y) = 0$$

primjetimo da x ne smije biti nula (u suprotnom iz prve jednakosti bi dobili $1=0$ # kontrad.

$$1 + x\sin y = 0$$
$$1 + x\cos y = 0$$

$$\sin y = \cos y$$

$y = \frac{\pi}{4} + 2k\pi, \ k \in \mathbb{Z}$
ili $y = \frac{5\pi}{4} + 2k\pi, \ k \in \mathbb{Z}$

Za $y = \frac{\pi}{4} + 2k\pi$ imamo

$$1 + x\frac{\sqrt{2}}{2} = 0$$

$$x = -\sqrt{2}$$

Za $y = \frac{5\pi}{4} + 2k\pi$ imamo

$$1 + x\left(-\frac{\sqrt{2}}{2}\right) = 0$$

$$x = \sqrt{2}$$

Stacionarne tačke su

$M_k\left(-\sqrt{2}; \frac{\pi}{4} + 2k\pi\right)$

i $N_k\left(\sqrt{2}; \frac{5\pi}{4} + 2k\pi\right)$.

Odredimo druge parcijalne izvode

$$\frac{\partial^2 f}{\partial x^2} = e^{y+x\sin y}\cdot \sin y + e^{y+x\sin y}\sin y = 2e^{y+x\sin y}\sin y$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{y+x\sin y}\cdot(1+x\cos y)(1+x\sin y) + e^{y+x\sin y}x\cos y$$

$$\frac{\partial^2 f}{\partial y^2} = x e^{y+x\sin y}\cdot(1+x\cos y)(1+x\cos y) + x e^{y+x\sin y}(-x\sin y)$$

Za tačke $M_k(-\sqrt{2}, \frac{\pi}{4}+2k\pi)$ imamo $1+x\sin y=0$ i $1+x\cos y=0$

pa je $A=2e^{\frac{\pi}{4}-\sqrt{2}\cdot\frac{\sqrt{2}}{2}}\cdot\frac{\sqrt{2}}{2} = \sqrt{2}\, e^{\frac{\pi}{4}-1}$, $B=-e^{\frac{\pi}{4}-1}$, $C=-\sqrt{2}\, e^{\frac{\pi}{4}-1}$

$$D=\begin{vmatrix} A & B \\ B & C\end{vmatrix} = AC-B^2 = -2e^{\frac{\pi}{2}-2} - e^{\frac{\pi}{2}-2} = -3e^{\frac{\pi}{2}-2} < 0$$

U tačkama $M_k(-\sqrt{2}; \frac{\pi}{4}+2k\pi)$ f-ja nema ekstrem.

Za tačke $N_k(\sqrt{2}, \frac{5\pi}{4}+2k\pi)$ imamo

$A=-\sqrt{2}\, e^{\frac{5\pi}{4}-1}$, $B=-e^{\frac{5\pi}{4}-1}$, $C=\sqrt{2}\, e^{\frac{5\pi}{4}-1}$

$$D=\begin{vmatrix} A & B \\ B & C\end{vmatrix} = AC-B^2 = -2e^{\frac{5\pi}{2}-2} - e^{\frac{5\pi}{2}-2} < 0$$

U tačkama $N_k(\sqrt{2}, \frac{5\pi}{4}+2k\pi)$ f-ja nema ekstrem.

(Tačke $M_k(-\sqrt{2}; \frac{\pi}{4}+2k\pi)$ i $N_k(\sqrt{2}; \frac{5\pi}{4}+2k\pi)$ su sedlaste tačke).

(#) Naći ekstreme f-je $z = x^3 + 4x^2y + xy^2 - 12xy - 3y^2$.

Rj.

$$\frac{\partial z}{\partial x} = 3x^2 + 8xy + y^2 - 12y$$

$$\frac{\partial z}{\partial y} = 4x^2 + 2xy - 12x - 6y$$

$$3x^2+8xy+y^2-12y=0$$
$$4x^2+2xy-12x-6y=0$$

$$3x^2+8xy+y^2-12y=0$$
$$4x(x-3)+2y(x-3)=0$$

$$3x^2+8xy+y^2-12y=0$$
$$(4x+2y)(x-3)=0$$

$$3x^2+8xy+y^2-12y=0$$
$4x+2y=0$ ili $x-3=0$

$y=-2x$ ili $x=3$

a) za $x=3$ imamo

$$3\cdot 9+8\cdot 3y+y^2-12y=0$$
$$y^2+12y+27=0$$
$$D=144-108$$
$$y_1=-9, \quad y_2=-3$$

b) $y=-2x$

$$3x^2+8x\cdot(-2x)+(-2x)^2-12(-2x)=0$$
$$3x^2-16x^2+4x^2+24x=0$$
$$-9x^2+24x=0$$
$$-3x(3x-8)=0$$
$$x_1=0, \quad x_2=\frac{8}{3}$$

Stacionarne tačke su $M_1(3,-9)$, $M_2(3,-3)$, $M_3(0,0)$, i $M_4\left(\frac{8}{3}, -\frac{16}{3}\right)$.

$$\frac{\partial^2 z}{\partial x^2} = 6x+8y$$

$$\frac{\partial^2 z}{\partial x \partial y} = 8x+2y-12$$

$$\frac{\partial^2 z}{\partial y^2} = 2x-6$$

Za $M_1(3,-9)$

$A=-54, \quad B=-6, \quad C=0$

$$D=\begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC-B^2 = -36<0$$

F-ja u tački M_1 nema ekstrem.

Za $M_2(3,-3)$

$A=-6,\ B=6,\ C=0$

$D=AC-B^2=-36<0$

F-ja u tački $M_2(3,-3)$ nema ekstrem.

Za $M_3(0,0)$

$A=0,\ B=-12,\ C=-6$

$D=AC-B^2=-144<0$

F-ja u tački $M_3(0,0)$ nema ekstrem

Za $M_4\left(\frac{8}{3},-\frac{16}{3}\right)$

$A=-\frac{80}{3},\ B=-\frac{4}{3},\ C=-\frac{2}{3}$

$D=AC-B^2=16>0$

F-ja u tački $M_4\left(\frac{8}{3},-\frac{16}{3}\right)$ ima ekstrem.

$A<0 \Rightarrow$ f-ja ima maksimum

$z_{MAX}\left(\frac{8}{3},-\frac{16}{3}\right)=\ldots=\frac{256}{9}$

(#) Izračunati dvostruki integral $I = \iint\limits_D xy\,dx\,dy$, gdje je D oblast ograničena linijama $xy = 1$, $x + y = \frac{5}{2}$.

Rj. Skiciramo oblast D

$xy = 1$

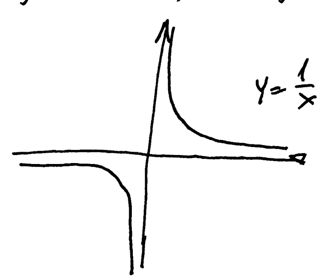
$y = \frac{1}{x}$ $\quad \mathcal{D}: x \in \mathbb{R} \setminus \{0\}$

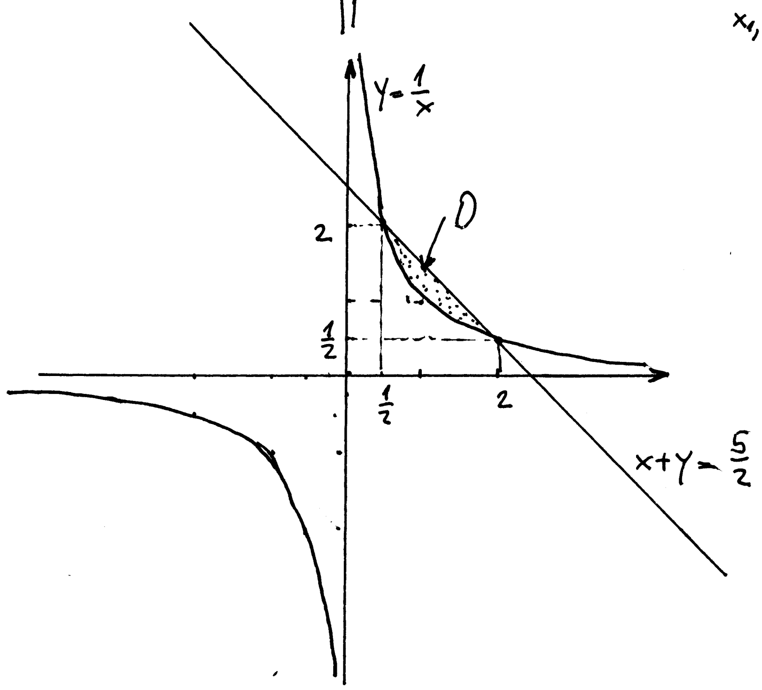
f-ja je neparna (simetrična u odnosu na 0)

ne sijeće y-osu, ne sijeće x-osu

$\lim\limits_{x \to 0+} \frac{1}{x} = +\infty$

$\lim\limits_{x \to \infty} \frac{1}{x} = 0$





Nađimo presječne tačke krive $xy = 1$ i prave $x + y = \frac{5}{2}$.

$$xy = 1$$
$$x + y = \frac{5}{2}$$

$$y = \frac{1}{x}$$
$$x + y = \frac{5}{2}$$

$$x + \frac{1}{x} = \frac{5}{2} \quad / \cdot x$$

$$x^2 - \frac{5}{2}x + 1 = 0 \quad / \cdot 2$$

$$2x^2 - 5x + 2 = 0$$

$$D = 25 - 16 = 9$$

$$x_{1,2} = \frac{5 \pm 3}{4} \qquad x_1 = \frac{2}{4} = \frac{1}{2}$$
$$x_2 = 2$$

$x_1 = \frac{1}{2} \Rightarrow y_1 = 2$

$x_2 = 2 \Rightarrow y = \frac{1}{2}$

$$D: \begin{cases} \frac{1}{2} < x < 2 \\ \frac{1}{x} < y < \frac{5}{2} - x \end{cases}$$

$$\iint\limits_D xy\,dx\,dy = \int_{\frac{1}{2}}^{2} x\,dx \int_{\frac{1}{x}}^{\frac{5}{2}-x} y\,dy =$$

$$= \int_{\frac{1}{2}}^{2} x \, \frac{1}{2} y^2 \Big|_{\frac{1}{x}}^{\frac{5}{2}-x} dx = \frac{1}{2}\int_{\frac{1}{2}}^{2} x\left(\left(\frac{5}{2} - x\right)^2 - \frac{1}{x^2}\right)dx = \frac{1}{2}\int_{\frac{1}{2}}^{2} x\left(\frac{25}{4} - 5x + x^2 - \frac{1}{x^2}\right)dx$$

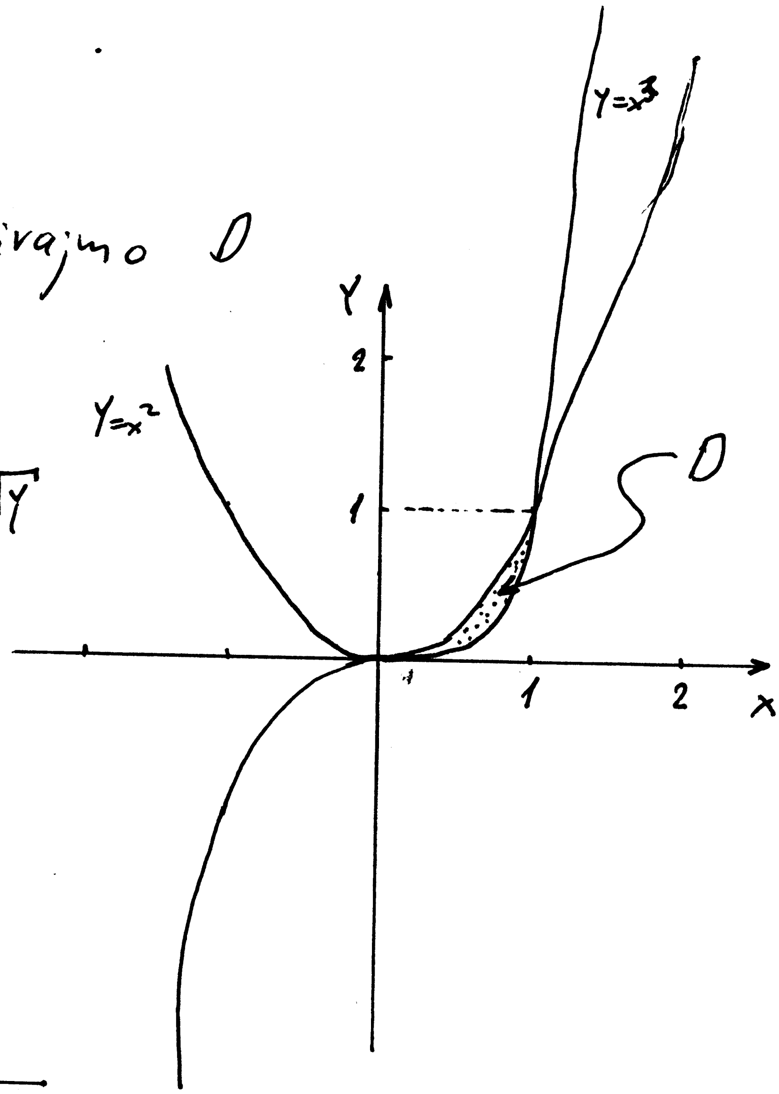
$$\frac{1}{2}\int_{\frac{1}{2}}^{2}\left(\frac{25}{4}x - 5x^2 + x^3 - \frac{1}{x}\right)dx = \int_{\frac{1}{2}}^{2}\left(\frac{1}{2}x^3 - \frac{5}{2}x^2 + \frac{25}{8}x - \frac{1}{2}\cdot\frac{1}{x}\right)dx = \ldots = \frac{165}{128} - \ln 2$$

rješenje

Izmjeniti poredak integracije u integralu

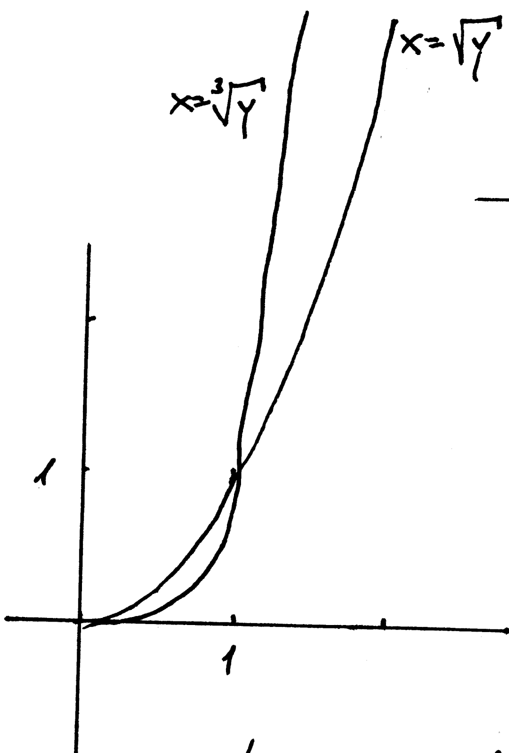
$$\int_0^1 dx \int_{x^3}^{x^2} f(x,y)\, dy .$$

Rj. $D: \begin{cases} 0 \le x \le 1 \\ x^3 \le y \le x^2 \end{cases}$ Skicirajmo D



$y = x^3$

$y = x^2$



Kako je

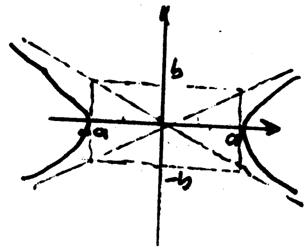
to je $D: \begin{cases} 0 \le y \le 1 \\ \sqrt{y} \le x \le \sqrt[3]{y} \end{cases}$

$$\int_0^1 dx \int_{x^3}^{x^2} f(x,y)\, dy = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} f(x,y)\, dx$$

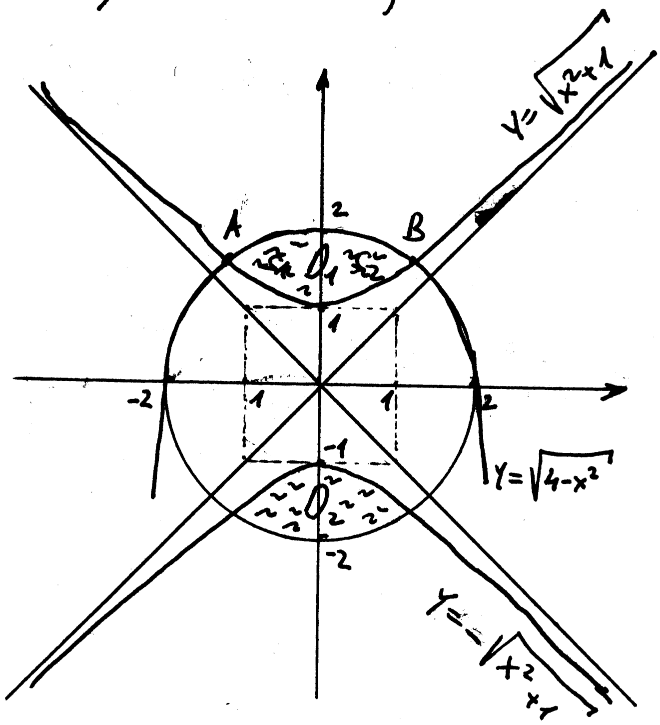
traženo rješenje

(#) Izračunati $I=\iint\limits_D dx\,dy$, ako je $D: y^2-x^2=1,\ x^2+y^2=4$.

Rj. Krive oblika $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ zovemo hiperbole i one su oblika



Skicirajmo naše dvije krive



$x^2+y^2=4$ je krug sa centrom u $(0,0)$ poluprečnika $r=2$

$D=D_1\cup D_2$

$$I=\iint\limits_D dx\,dy=\iint\limits_{D_1\cup D_2} dx\,dy=2\iint\limits_{D_1} dx\,dy$$

$y^2=4-x^2 \qquad y^2=x^2+1$

$y=\pm\sqrt{4-x^2} \qquad y=\pm\sqrt{x^2+1}$

Nađimo presečnu tačku krivih $y=\sqrt{x^2+1}$ i $y=\sqrt{4-x^2}$

$\sqrt{x^2+1}=\sqrt{4-x^2}\ /^2$

$x^2+1=4-x^2$

$2x^2-3=0$

$2x^2=3$

$x^2=\frac{3}{2}$

$x_1=-\sqrt{\frac{3}{2}}$

$x_2=\sqrt{\frac{3}{2}}$

$x_1=-\sqrt{\frac{3}{2}} \Rightarrow y=\sqrt{\frac{3}{2}+1}=\sqrt{\frac{5}{2}}$

Presečne tačke su $A\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$ i $B\left(\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$.

Primetimo da je oblast D_1 simetrična

$$\iint\limits_{D_1} dx\,dy=\iint\limits_{S_1} dx\,dy+\iint\limits_{S_2} dx\,dy=2\iint\limits_{S_2} dx\,dy=2\int_0^{\sqrt{\frac{3}{2}}}dx\int_{\sqrt{x^2+1}}^{\sqrt{4-x^2}}dy=2\int_0^{\sqrt{\frac{3}{2}}}\left(\sqrt{4-x^2}-\sqrt{x^2+1}\right)dx$$

$\int\sqrt{4-x^2}\,dx=2\arcsin\frac{x}{2}+\frac{x\sqrt{4-x^2}}{2}$ (sami)

(metodom Ostrogradskog)

$\int\sqrt{x^2+1}\,dx=\int\frac{x^2+1}{\sqrt{x^2+1}}dx=(ax+b)\sqrt{x^2+1}+\lambda\int\frac{dx}{\sqrt{x^2+1}}\quad /'$

$$\frac{x^2+1}{\sqrt{x^2+1}} = a\sqrt{x^2+1} + (ax+b)\frac{2x}{2\sqrt{x^2+1}} + \lambda \frac{1}{\sqrt{x^2+1}} \quad / \cdot \sqrt{x^2+1}$$

$$x^2+1 = a(x^2+1) + ax^2 + bx + \lambda$$

$$2a = 1 \quad \Rightarrow \quad a = \frac{1}{2}$$

$$b = 0$$

$$a + \lambda = 1 \qquad \lambda = \frac{1}{2}$$

$$\int \sqrt{x^2+1}\, dx = \frac{1}{2} x \sqrt{x^2+1} + \frac{1}{2} \int \frac{dx}{\sqrt{x^2+1}}$$

$$\int \sqrt{x^2+1}\, dx = \frac{1}{2} x \sqrt{x^2+1} + \frac{1}{2} \ln|x + \sqrt{x^2+1}| + C$$

$$\int_0^{\sqrt{\frac{3}{2}}} \sqrt{4-x^2}\, dx = 2 \arcsin \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} \quad \text{(sami)}$$

$$\int_0^{\sqrt{\frac{3}{2}}} \sqrt{x^2+1}\, dx = \frac{1}{2} \cdot \sqrt{\frac{3}{2}} \overbrace{\sqrt{\frac{3}{2}+1}}^{\sqrt{\frac{5}{2}}} + \frac{1}{2} \ln\left|\sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}+1}\right| = \frac{\sqrt{15}}{4} + \frac{1}{2} \ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

$$I = \iint_D dx\,dy = 2 \iint_{D_1} dx\,dy = 2 \cdot 2 \iint_{S_2} dx\,dy = 4\left(2 \arcsin \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} - \right.$$

$$\left. \frac{\sqrt{15}}{4} - \frac{1}{2} \ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right| \right) = 8 \arcsin \frac{\sqrt{6}}{4} - 2 \ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

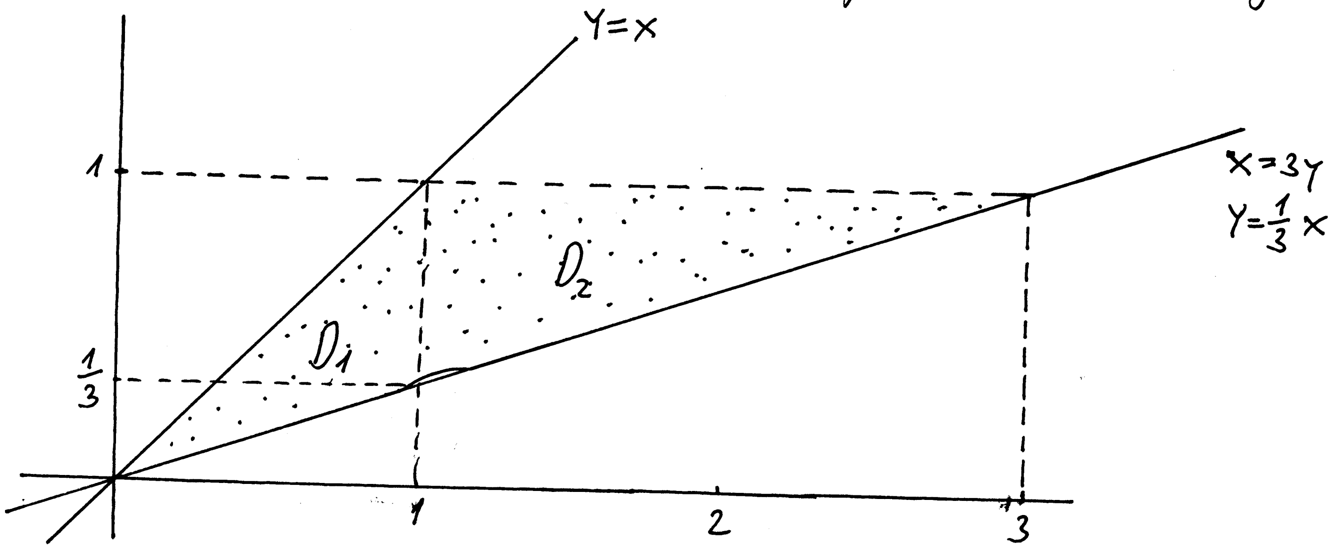
traženo vjerenje.

Izmjeniti poredak integracije u integralu

$$I = \int_0^1 dy \int_y^{3y} f(x,y)\,dx \,.$$

Rj.

$x = 3y$; $x = y$ su prave. Skicirajmo oblast integracije



$$D: \begin{cases} 0 \le y \le 1 \\ y \le x \le 3y \end{cases} = D_1 \cup D_2$$

$$D_1: \begin{cases} 0 \le x \le 1 \\ \frac{1}{3}x \le y \le x \end{cases}$$

$$D_2: \begin{cases} 1 \le x \le 3 \\ \frac{1}{3}x \le y \le 1 \end{cases}$$

$$\int_0^1 dy \int_y^{3y} f(x,y)\,dx = \int_0^1 dx \int_{\frac{1}{3}x}^{x} f(x,y)\,dy + \int_1^3 dx \int_{\frac{1}{3}x}^{1} f(x,y)\,dy$$

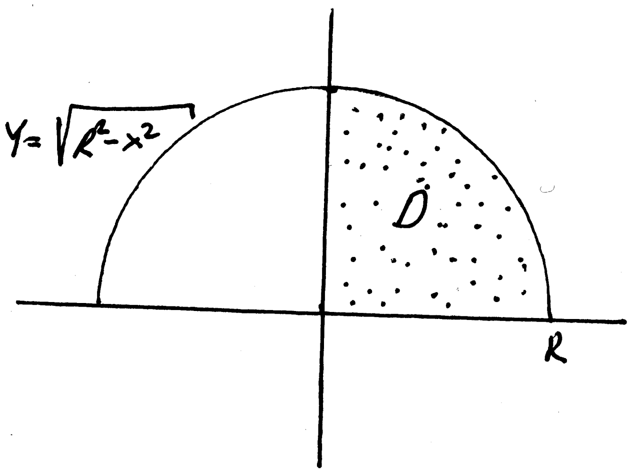
(#) Dati dvostruki integral $\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy$

iz pravougaonih koordinata transformisati na polarne koordinate.

Rj. Oblast integracije D prema postavci zadatka je

$$D: \begin{cases} 0 \le x \le R \\ 0 \le y \le \sqrt{R^2-x^2} \end{cases}$$

Skicirajmo oblast D.



$$y^2 = R^2 - x^2$$

$$x^2 + y^2 = R^2$$

Polarne koordinate glase

$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

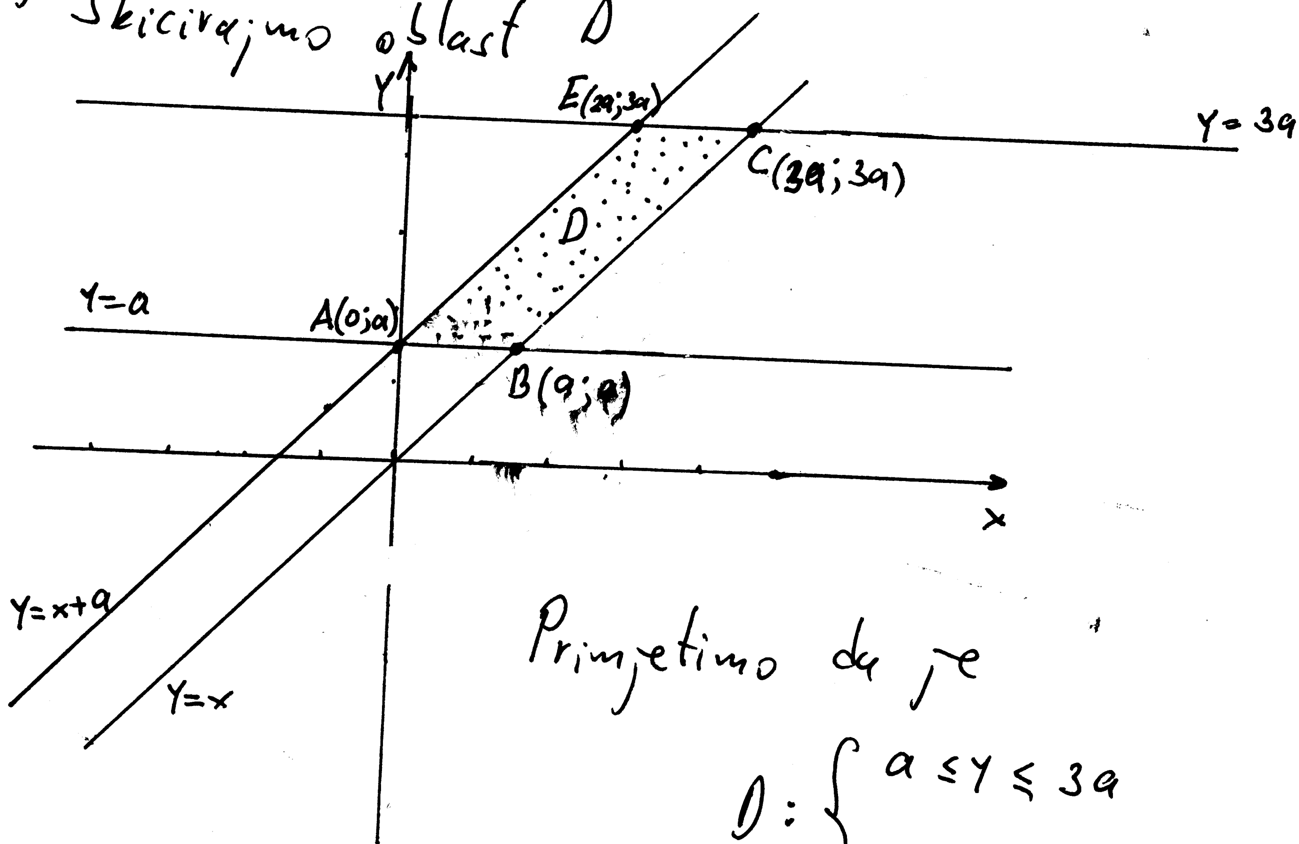
$$dx\,dy = r\,dr\,d\varphi$$

$$D \xrightarrow{\text{transformise}} D': \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$

$$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy = \int_0^R r\,dr \int_0^{\pi/2} f(r\cos\varphi, r\sin\varphi)\,d\varphi$$

(#) Izračunati $I=\iint_D (x^2+y^2)\,dx\,dy$ gdje je D paralelogram sa stranicama $y=x$, $y=x+a$, $y=a$, $y=3a$ $(a>0)$.

Rj: Skicirajmo oblast D



Primjetimo da je

$$D:\begin{cases} a\le y\le 3a\\ y-a\le x\le y\end{cases}$$

$$\iint_D (x^2+y^2)\,dx\,dy=\int_a^{3a}dy\int_{y-a}^{y}(x^2+y^2)\,dx=\int_a^{3a}\left(\frac{1}{3}x^3\Big|_{y-a}^{y}+y^2x\Big|_{y-a}^{y}\right)dy$$

$$=\int_a^{3a}\left[\frac{1}{3}\left(y^3-(y-a)^3\right)+y^2\left(y-(y-a)\right)\right]dy=\frac{1}{3}\int_a^{3a}y^3\,dy-\frac{1}{3}\int_a^{3a}(y-a)^3\,d(y-a)$$

$$+a\int_a^{3a}y^2\,dy=\frac{1}{3}\cdot\frac{1}{4}y^4\Big|_a^{3a}-\frac{1}{3}\cdot\frac{1}{4}(y-a)^4\Big|_a^{3a}+\frac{a}{3}y^3\Big|_a^{3a}=\frac{20}{3}a^4-\frac{4}{3}a^4+\frac{26}{3}a^4=14a^4$$

traženo rješenje

(#) Izračunati dvostruki integral dat u polarnim koordinatama

$$I=\iint_D \rho \sin\varphi \, d\rho \, d\varphi$$ gdje je oblast D

a) kružni sektor, ograničen linijama $\rho=a$, $\varphi=\frac{\pi}{2}$ i $\varphi=\pi$

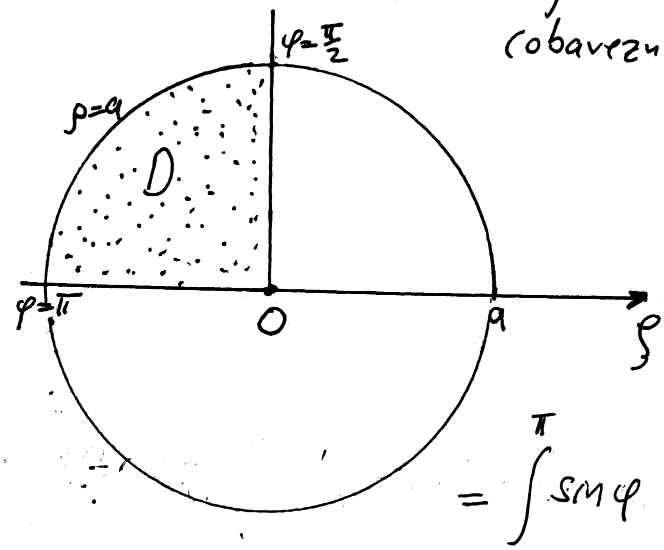
b) polukrug $\rho\le 2a\cos\varphi$, $0\le\varphi\le\frac{\pi}{2}$

c) oblast između linija $\rho=2+\cos\varphi$ i $\rho=1$.

(obavezno nacrtati izgled oblasti D)

Rj.

a)



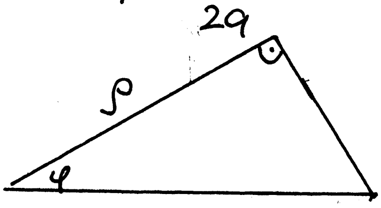
$$I=\iint_D \rho\sin\varphi\, d\rho\, d\varphi=$$

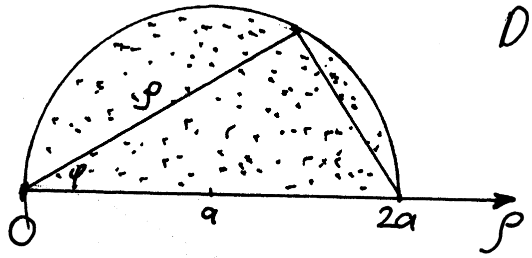
$$=\int_{\frac{\pi}{2}}^{\pi}\sin\varphi\, d\varphi\int_0^a \rho\, d\rho=$$

$$=\int_{\frac{\pi}{2}}^{\pi}\sin\varphi\,\frac{\rho^2}{2}\Big|_0^a d\varphi=\frac{a^2}{2}\left(-\cos\varphi\Big|_{\frac{\pi}{2}}^{\pi}\right)=\frac{a^2}{2}$$

b) $\rho=2a\cos\varphi$

$\cos\varphi=\dfrac{\rho}{2a}$





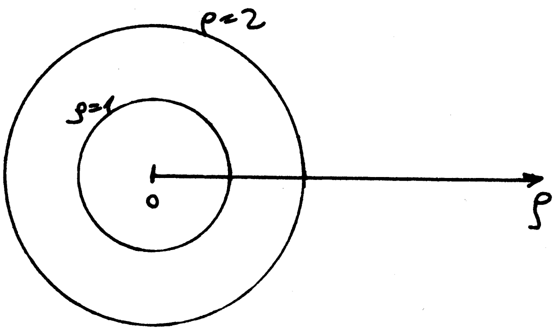
$$D:\begin{cases}0\le\rho\le 2a\cos\varphi\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}$$

$$I=\iint_D\rho\sin\varphi\,d\rho\,d\varphi=\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\int_0^{2a\cos\varphi}\rho\,d\rho=\int_0^{\frac{\pi}{2}}\frac{1}{2}\rho^2\Big|_0^{2a\cos\varphi}\sin\varphi\,d\varphi=$$

$$=\frac{1}{2}\cdot 4a^2\int_0^{\pi/2}\sin\varphi\cos^2\varphi\,d\varphi=\left|\begin{array}{l}\cos\varphi=t\\ -\sin\varphi\,d\varphi=dt\\ \varphi\Big|_0^{\frac{\pi}{2}}\Rightarrow t\Big|_1^0\end{array}\right|=2a^2\left(-\int_1^0 t^2\,dt\right)=$$

$$=2a^2\int_0^1 t^2\,dt=2a^2\cdot\frac{t^3}{3}\Big|_0^1=\frac{2}{3}a^2$$ traženo rješenje

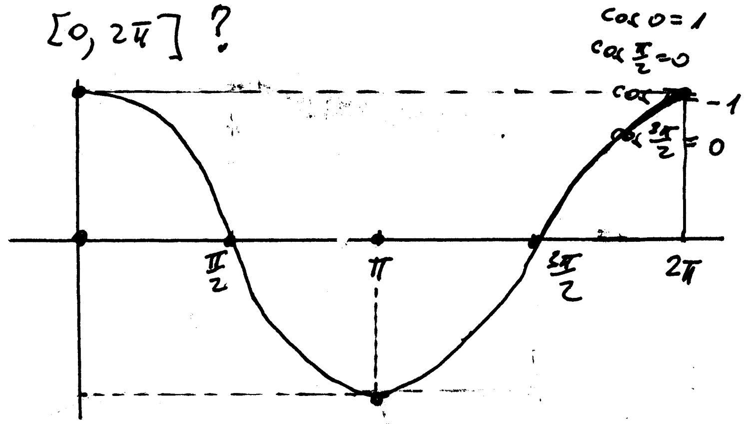
c) Linije $\rho=1$ i $\rho=2$ nije teško nacrtati



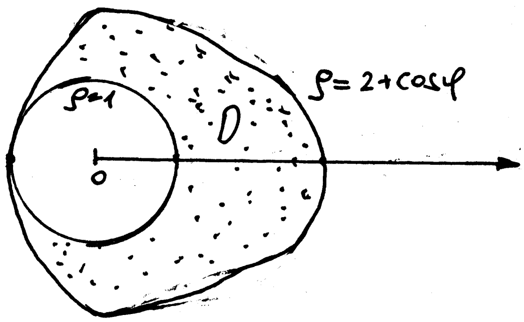
Problem predstavlja linija

$$\rho = 2+\cos\varphi$$

Kako izgleda $\cos\varphi$ na intervalu $[0, 2\pi]$?



Ako liniji $\rho=2$ dodamo $\cos\varphi$ imamo otprilike sljedeću sliku:



$$D: \begin{cases} 1 \le \rho \le 2+\cos\varphi \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$I=\iint_D \rho\sin\varphi\, d\rho\, d\varphi = \int_0^{2\pi}\sin\varphi\, d\varphi \int_1^{2+\cos\varphi}\rho\, d\rho = \int_0^{2\pi} \frac{\rho^2}{2}\Big|_1^{2+\cos\varphi}\sin\varphi\, d\varphi =$$

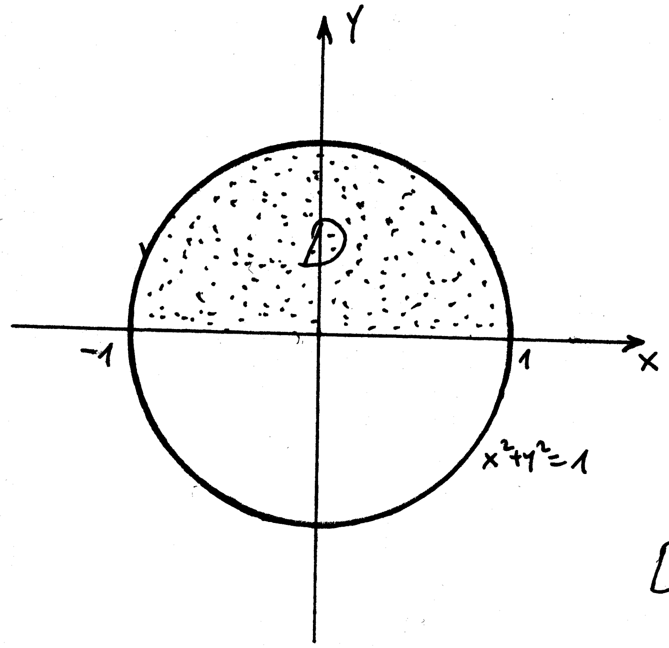
$$= \frac{1}{2}\int_0^{2\pi}\left((2+\cos\varphi)^2-1^2\right)\sin\varphi\, d\varphi = \frac{1}{2}\int_0^{2\pi}(4+4\cos\varphi+\cos^2\varphi-1)\sin\varphi\, d\varphi$$

$$= -\frac{1}{2}\int_0^{2\pi}(\cos^2\varphi+4\cos\varphi+3)\, d\cos\varphi = \left(-\frac{1}{2}\right)\left(\frac{\cos^3\varphi}{3}\Big|_0^{2\pi} + 4\,\frac{\cos^2\varphi}{2}\Big|_0^{2\pi} + 3\cos\varphi\Big|_0^{2\pi}\right)$$

$$= \left(-\frac{1}{2}\right)\left(\frac{1}{3}(1-1)+2(1-1)+3(1-1)\right) = 0$$ traženo rješenje

(#) Izračunati integral $I=\iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$, ako je D oblast data sa: $x^2+y^2\le 1$, $y\ge 0$.

Rj. Skicirajmo oblast D



$$D:\begin{cases} -1\le x\le 1\\ 0\le y\le \sqrt{1-x^2}\end{cases}$$

Uvedimo polarne koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$D \xrightarrow{\text{transformacije}} D'$, $\quad D':\begin{cases} 0\le r\le 1\\ 0\le \varphi\le \pi\end{cases}$

$1-x^2-y^2=1-(x^2+y^2)=1-r^2$

$1+x^2+y^2=1+r^2$

$$I=\iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy=\iint\limits_{D'}\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr\,d\varphi=\int_0^{\pi}d\varphi\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr$$

Izračunajmo posebno drugi integral

$$\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr=\int_0^1\frac{\sqrt{1-r^2}}{\sqrt{1+r^2}}\,r\,dr=\int_0^1\frac{1-r^2}{\sqrt{(1+r^2)(1-r^2)}}\cdot r\,dr=\int_0^1\frac{r}{\sqrt{1-r^4}}\,dr-\int_0^1\frac{r^3}{\sqrt{1-r^4}}\,dr$$

$$\int_0^1\frac{r}{\sqrt{1-r^4}}\,dr=\left|\begin{matrix} r^2=t\\ 2r\,dr=dt\\ r\,dr=\frac{1}{2}dt\end{matrix}\right| \; (r|_0^1\Rightarrow t|_0^1) =\frac{1}{2}\int_0^1\frac{dt}{\sqrt{1-t^2}}=\frac{1}{2}\arcsin t\Big|_0^1=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

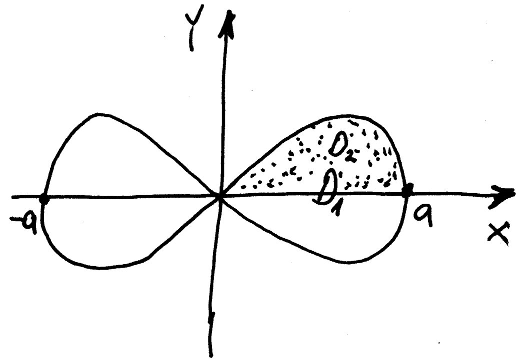
$$\int_0^1\frac{r^3}{\sqrt{1-r^4}}\,dr=\left|\begin{matrix} 1-r^4=s^2 & r^3dr=-\frac{1}{2}s\,ds\\ -4r^3dr=2s\,ds & r|_0^1\Rightarrow s|_1^0\end{matrix}\right|=-\frac{1}{2}\int_1^0\frac{s\,ds}{\sqrt{s^2}}=\frac{1}{2}$$

$$I=\int_0^{\pi}d\varphi\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr=\varphi\Big|_0^{\pi}\cdot\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{\pi^2}{4}-\frac{\pi}{2}$$ traženo rješenje

(#) Izračunati dvostruki integral $\iint_D dx\,dy$, ako je D oblast ograničena lemniskatom

$$(x^2+y^2)^2 = a^2(x^2-y^2).$$

Rj. Lemniskata grafički izgleda ovako.



Pronađimo presječne tačke lemniskate sa x-osom: $y=0$

$x^4 = a^2x^2 \Rightarrow x^4 - a^2x^2 = 0$

$x^2(x^2-a^2) = 0$

$x_1 = 0,\ x_2 = a,\ x_3 = -a$

Primjetimo da se površina oblasti D računa po formuli $P = \iint_D dx\,dy$. Naša oblast D je simetrična u odnosu na y-osu pa je $\iint_D dx\,dy = 2\iint_{D_1} dx\,dy$.

Oblast D_1 je simetrična u odnosu na x-osu.

$$\iint_D dx\,dy = 4\iint_{D_2} dx\,dy$$

uvedimo polarne koordinate $x = r\cos\varphi$, $y = r\sin\varphi$, $dx\,dy = r\,dr\,d\varphi$

$$(x^2+y^2)^2 = a^2(x^2-y^2)$$

$$(r^2)^2 = a^2(r^2\cos^2\varphi - r^2\sin^2\varphi) \quad /:r^2 \ (r\neq 0)$$

$$r^2 = a^2(\cos^2\varphi - \sin^2\varphi)$$

$$r^2 = a^2\cdot\cos 2\varphi \Rightarrow r = a\sqrt{\cos 2\varphi}$$

(primjetimo da za $\varphi > \frac{\pi}{4}$ r nije definisano)

$$D_2: \begin{cases} 0 < \varphi < \frac{\pi}{4} \\ 0 < r < \sqrt{a^2\cos 2\varphi} \end{cases}$$

$$\iint_D dx\,dy = 4\iint_{D_2} r\,dr\,d\varphi = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^{a\sqrt{\cos 2\varphi}} r\,dr = 4\int_0^{\frac{\pi}{4}} \frac{1}{2}r^2\Big|_0^{a\sqrt{\cos 2\varphi}} d\varphi = \frac{4}{2}\int_0^{\frac{\pi}{4}} a^2\cos 2\varphi\,d\varphi =$$

$$= 2a^2\cdot\frac{1}{2}\sin 2\varphi\Big|_0^{\frac{\pi}{4}} = a^2\left(\sin\frac{\pi}{2} - 0\right) = a^2 \quad \text{traženo rješenje}$$

(#) Izračunati dvostruki integral $I=\iint\limits_D (x^2+y^2)\,dx\,dy$ gdje je

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le \frac{2}{3}(x+2y)\}$.

Rj. Odredimo šta je oblast D.

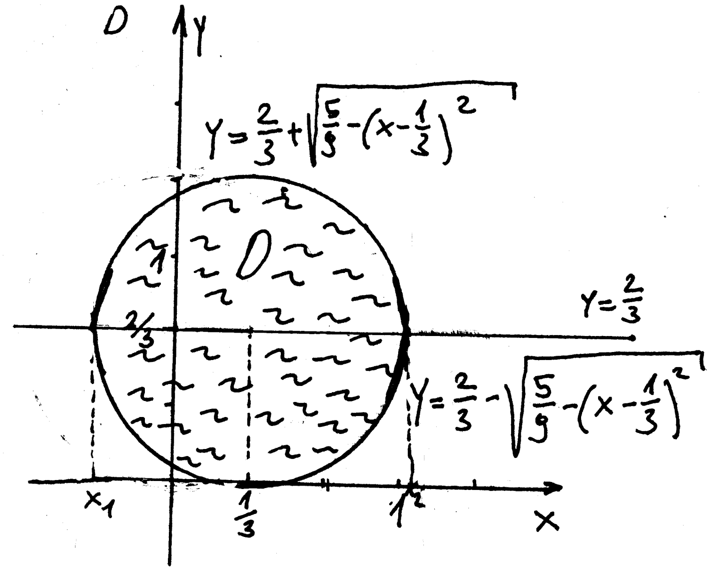
$$x^2+y^2\le\frac{2}{3}(x+2y)$$

$$x^2+y^2\le\frac{2}{3}x+\frac{4}{3}y$$

$$x^2-\frac{2}{3}x+y^2-\frac{4}{3}y\le 0$$

$$x^2-2\cdot x\cdot\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+y^2-2\cdot y\cdot\frac{2}{3}+\frac{4}{9}-\frac{4}{9}\le 0$$

$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2\le\frac{5}{9}$$



D predstavlja unutrašnjost kruga s centrom u tački $\left(\frac{1}{3},\frac{2}{3}\right)$ poluprečnika $r=\frac{\sqrt{5}}{3}\approx 0{,}74$

$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2=\frac{5}{9}$$

$$y=\frac{2}{3}$$

$$\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$$

$$x-\frac{1}{3}=\pm\frac{\sqrt{5}}{3}$$

$$x_{1,2}=\frac{1\pm\sqrt{5}}{3}$$

$$\left(y-\frac{2}{3}\right)^2=\frac{5}{9}-\left(x-\frac{1}{3}\right)^2$$

$$y=\frac{2}{3}\pm\sqrt{\frac{5}{9}-\left(x-\frac{1}{3}\right)^2}$$

I način: klasičan način

Nađimo presječnu tačku kruga i prave $y=\frac{2}{3}$

$$I=\iint\limits_D (x^2+y^2)\,dx\,dy=\int_{\frac{1-\sqrt{5}}{3}}^{\frac{1+\sqrt{5}}{3}}\left[\int_{\frac{2}{3}-\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}^{\frac{2}{3}+\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}(x^2+y^2)\,dy\right]dx=\ldots$$

NA KLASIČAN NAČIN OVO JE TEŠKO URADITI

II način: Uvedimo neku smjenu promjenljivih. Kako je dat krug uvedimo polarne koordinate.

$$dx\,dy=|J|\,dr\,d\varphi \quad \text{(Jakobijan)}$$

$x=a+r\cos\varphi$, $y=b+r\sin\varphi$ tj. $x=\frac{1}{3}+r\cos\varphi$, $y=\frac{2}{3}+r\sin\varphi$

$0\le\varphi\le 2\pi$

$0\le r\le\frac{\sqrt{5}}{3}$

ove vrijednosti čitamo sa slike

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

$dx\,dy=r\,dr\,d\varphi$.

$$x^2+y^2=\left(\frac{1}{3}+r\cos\varphi\right)^2+\left(\frac{2}{3}+r\sin\varphi\right)^2=\frac{1}{9}+\frac{2}{3}r\cos\varphi+r^2\cos^2\varphi+\frac{4}{9}+\frac{4}{3}r\sin\varphi+r^2\sin^2\varphi=$$

$$=\frac{5}{9}+r^2+\frac{2}{3}r\cos\varphi+\frac{4}{3}r\sin\varphi$$

$$I=\iint\limits_D (x^2+y^2)\,dx\,dy=\iint\limits_{D'}\left(\frac{5}{9}+r^2+\frac{2}{3}r(\cos\varphi+2\sin\varphi)\right)r\,dr\,d\varphi=\iint\limits_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi+$$

$$+\frac{2}{3}\iint\limits_{D'}r^2(\cos\varphi+2\sin\varphi)\,dr\,d\varphi,\quad \iint\limits_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{\sqrt{5}}{3}}\left(\frac{5}{9}r+r^3\right)dr\right]d\varphi=2\pi\cdot\left(\frac{5}{9}\cdot\frac{1}{2}r^2\Big|_0^{\frac{\sqrt{5}}{3}}+\right.$$

$$+\frac{1}{4}r^4\Big|_0^{\frac{\sqrt{5}}{3}}\Big) = 2\pi\left(\frac{5}{9\cdot 2}\cdot\frac{5}{9}+\frac{1}{4}\cdot\frac{5\cdot 5}{9\cdot 9}\right) = \pi\left(\frac{5^2}{9^2}+\frac{1}{2}\cdot\frac{5^2}{9^2}\right) = \frac{3}{2}\,\frac{5^2}{9^2}\pi = \frac{25}{54}\pi$$

$$\iint_{D'} r^2(\cos\varphi + 2\sin\varphi)\,dr\,d\varphi = \int_0^{\sqrt{5}/3} r^2\left[\int_0^{2\pi}(\cos\varphi + 2\sin\varphi)\,d\varphi\right] = \frac{r^3}{3}\Big|_0^{\sqrt{5}/3}\left(\sin\varphi\Big|_0^{2\pi} - 2\cos\varphi\Big|_0^{2\pi}\right) = 0$$

Prema tome $\iint_D (x^2+y^2)\,dx\,dy = \frac{25}{54}\pi$

Dati dvostruki integral $\int_{\frac{R}{2}}^{2R} dy \int_{0}^{\sqrt{2Ry-y^2}} f(x,y)\,dx$

iz pravougaonih koordinata transformisati na polarne koordinate.

Rj: Skicirajmo oblast integracije.

Iz postavke vidimo da je x ograničen sa pravom $x=0$ i krivom $x=\sqrt{2Ry-y^2}$

$$D:\begin{cases} 0 \le x \le \sqrt{2Ry-y^2} \\ 2R \le y \le R/2 \end{cases}$$

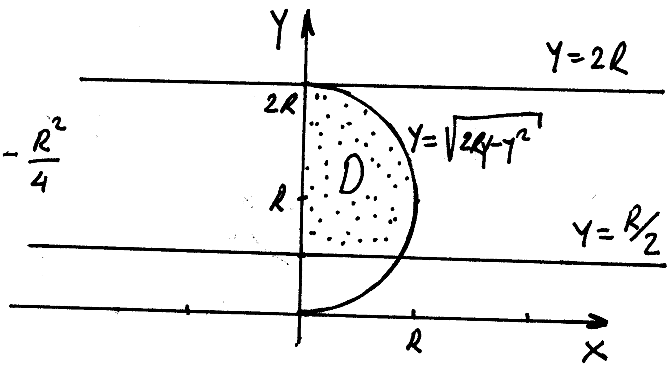
$x^2 = 2Ry - y^2$

$x^2 + y^2 - 2\cdot y\cdot R + R^2 - R^2 = 0$

$x^2 + (y-R)^2 = R^2$

krug sa centrom u tački $(0,R)$ poluprečnika R.

$2R\cdot\frac{R}{2} - \frac{R^2}{4}$



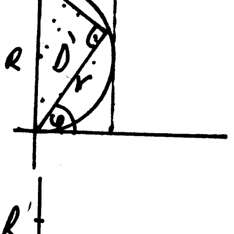
Polarne koordinate glase

$x = r\cos\varphi$

$y = r\sin\varphi$

$dx\,dy = r\,dr\,d\varphi$

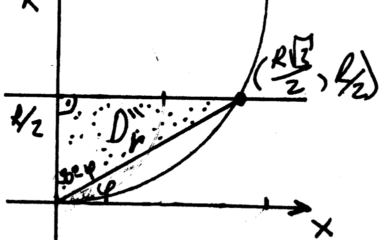
Da bi došli do ideje kako opisati oblast D posmatrajmo sljedeće "jednostavnije" oblasti D' i D'':



$\cos(90^\circ-\varphi) = \frac{r}{2R} \Rightarrow r = 2R\sin\varphi$

$\cos(90^\circ-\varphi) = \sin\varphi$

$$D':\begin{cases} 0 \le r \le 2R\sin\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$



$\cos(90^\circ-\varphi) = \frac{R/2}{r}$

$\sin\varphi = \frac{R}{2r}$

$2r = \frac{R}{\sin\varphi} \Rightarrow r = \frac{R}{2\sin\varphi}$

$$D'':\begin{cases} 0 \le r \le \frac{R}{2\sin\varphi} \\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{2} \end{cases}$$

Sad nije teško vidjeti da će oblast D opisana pomoću polarnih koordinata postati

$$D: \begin{cases} \frac{R}{2\sin\varphi} \leq r \leq 2R\sin\varphi \\ \frac{\pi}{6} \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

Prema tome

$$\int_{R/2}^{2R} dy \int_{0}^{\sqrt{2Ry-y^2}} f(x,y)\,dx = \int_{\pi/6}^{\pi/2} d\varphi \int_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} f(r\cos\varphi, r\sin\varphi)\, r\, dr$$

Izračunati integral $I=\int\limits_0^{\sqrt{3}/2} dx \int\limits_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy$.

Rj: Pokušajmo prvo skicirati oblast integracije D. Primjetimo da se u drugom integralu pojavljuju f-je $y=\sqrt{1-x^2}$ i $y=1-\sqrt{1-x^2}$. Nacrtajmo ih.

$y=\sqrt{1-x^2}$

$y^2=1-x^2$

$x^2+y^2=1$

krug sa centrom u $C(0,0)$ poluprečnika $r=1$

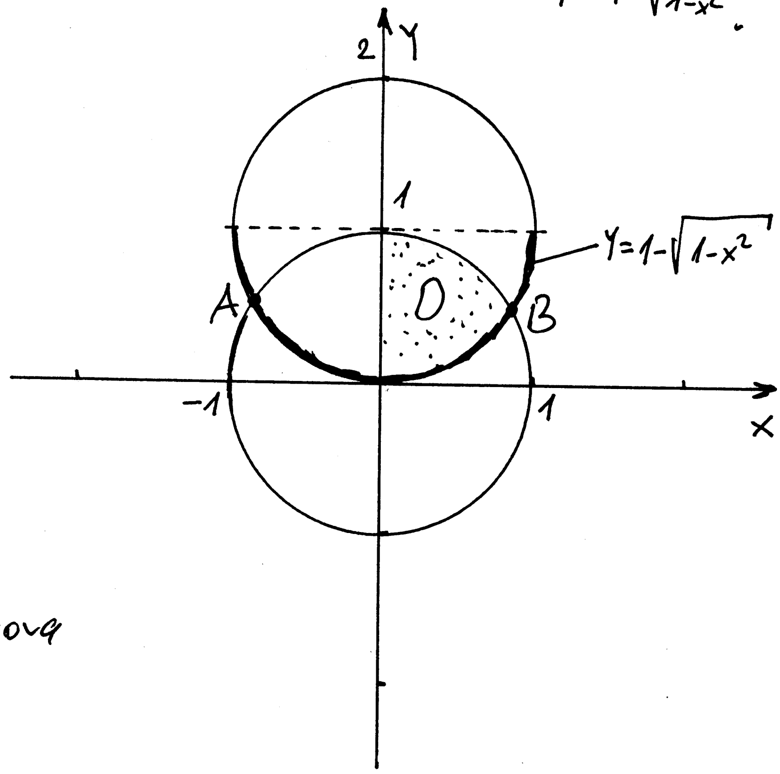
$y=1-\sqrt{1-x^2}$

$y-1=-\sqrt{1-x^2}$

$(y-1)^2=1-x^2$

$x^2+(y-1)^2=1$

krug sa centrom u $C(0,1)$ poluprečnika $r=1$



Pronađimo tačke presjeka ovih krugova

$$\frac{\begin{array}{c} x^2+y^2=1 \\ x^2+(y-1)^2=1 \end{array}}{\begin{array}{c} x^2=1-y^2 \\ x^2+(y-1)^2=1 \end{array}}$$

$1-y^2+(y-1)^2=1$

$1-y^2+y^2-2y+1=1$

$1-2y=0$

$2y=1$

$y=\frac{1}{2}$

$x^2+y^2=1$

$x^2=1-y^2$

$x^2=1-\frac{1}{4}$

$x^2=\frac{3}{4}$

$x_{1,2}=\pm\frac{\sqrt{3}}{2}$

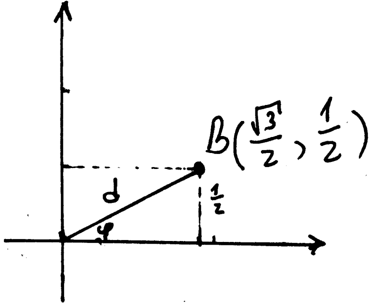
Tačke presjeka su $A(-\frac{\sqrt{3}}{2},\frac{1}{2})$ i $B(\frac{\sqrt{3}}{2},\frac{1}{2})$

Sad možemo konačno nacrtati oblast integracije D.

$$I=\int\limits_0^{\sqrt{3}/2} dx \int\limits_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy = \iint\limits_D \sqrt{x^2+y^2}\, dx\, dy$$

Oblast D ćemo podjeliti na dva dijela D_1 i D_2 pa ćemo imati

$$\iint\limits_D \sqrt{x^2+y^2}\, dx\, dy = \iint\limits_{D_1} \sqrt{x^2+y^2}\, dx\, dy + \iint\limits_{D_2} \sqrt{x^2+y^2}\, dx\, dy.$$

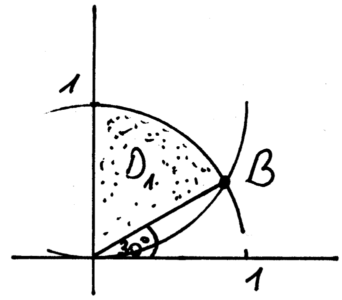


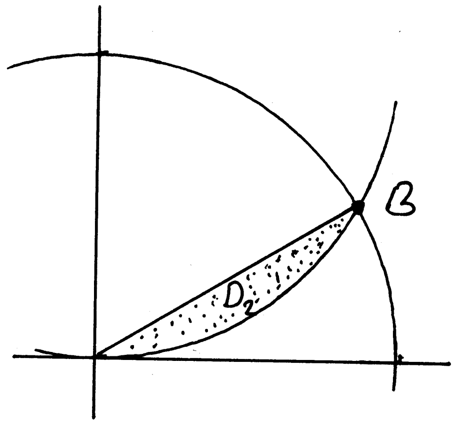
$$d = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

$$\sin\varphi = \frac{\frac{1}{2}}{1}$$

$$\sin\varphi = \frac{1}{2}$$

$$\varphi = 30^\circ$$

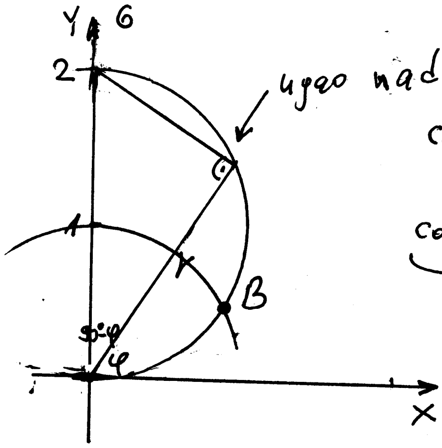




$$\iint_{D_1} \sqrt{x^2+y^2}\, dx\, dy = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x = r\cos\varphi \\ y = r\sin\varphi \\ dx\,dy = r\,dr\,d\varphi \end{array} \right.$$

$$D_1 \xrightarrow{\text{transformiše}} D_1' : \begin{cases} 0 \le r \le 1 \\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{2} \end{cases} \Bigg| = \iint_{D_1'} \sqrt{r^2}\, r\,dr\,d\varphi = \int_0^1 r^2\,dr \int_{\frac{\pi}{6}}^{\pi/2} d\varphi =$$

$$= \varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cdot \frac{1}{3} r^3 \Big|_0^1 = \frac{1}{3}\left(\frac{3\pi}{6} - \frac{\pi}{6}\right) = \frac{1}{3}\cdot\frac{2\pi}{6} = \frac{1}{3}\cdot\frac{\pi}{3} = \frac{\pi}{9}$$



ugao nad prečnikom

$$\cos(90^\circ - \varphi) = \frac{r}{2}$$

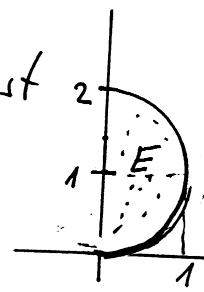
$$\underbrace{\cos 90^\circ \cos\varphi + \sin 90^\circ \sin\varphi}_{\cos(90^\circ - \varphi)} = \frac{r}{2}$$

$$\sin\varphi = \frac{r}{2}$$

$$r = 2\sin\varphi$$

Prema tome pomoćna oblast E ima granice

$$E: \begin{cases} 0 \le r \le 2\sin\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$



Odavde možemo vidjeti polarne granice za D_2

$$\iint_{D_2} \sqrt{x^2+y^2}\, dx\, dy = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x = r\cos\varphi \\ y = r\sin\varphi \\ dx\,dy = r\,dr\,d\varphi \end{array} \right. \quad D_2 \xrightarrow{\text{transformiše}} D_2' : \begin{cases} 0 \le r \le 2\sin\varphi \\ 0 \le \varphi \le \frac{\pi}{6} \end{cases} \Bigg|$$

$$= \iint_{D_2'} r^2\,dr\,d\varphi = \int_0^{\frac{\pi}{6}} d\varphi \int_0^{2\sin\varphi} r^2\,dr = \int_0^{\frac{\pi}{6}} \frac{1}{3} r^3 \Big|_0^{2\sin\varphi} d\varphi = \frac{8}{3}\int_0^{\frac{\pi}{6}} \sin^3\varphi\, d\varphi = \underset{\text{za vježbu}}{\dots} = -\sqrt{3} + \frac{16}{9}$$

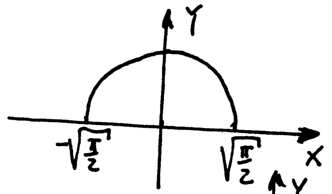
Prema tome $I = \frac{\pi}{9} + \frac{16}{9} - \sqrt{3} = \frac{\pi + 16}{9} - \sqrt{3}$ traženo rješenje

Izračunati dvostruki integral

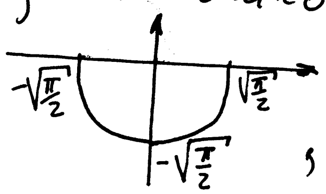
$$I=\int_0^{\sqrt{\frac{\pi}{2}}}dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy.$$

Rj. Oblast integracije D je $D:\begin{cases}0\le x\le\sqrt{\frac{\pi}{2}}\\ -\sqrt{\frac{\pi}{2}-x^2}\le y\le\sqrt{\frac{\pi}{2}-x^2}\end{cases}$

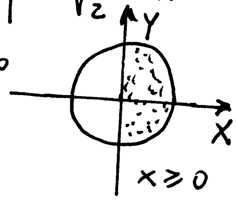
Znamo da f-ja $y=\sqrt{\frac{\pi}{2}-x^2}$ izgleda ovako



dok f-ja $y=-\sqrt{\frac{\pi}{2}-x^2}$ izgleda

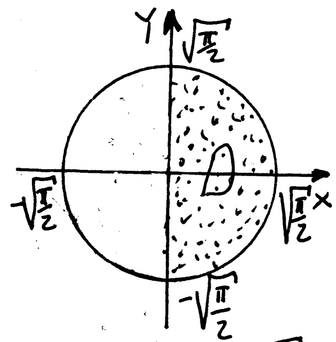


, sve zajedno $\Rightarrow$



.

Ove dvije f-je se dobiju iz f-je $y^2=\frac{\pi}{2}-x^2$ tj. $x^2+y^2=\frac{\pi}{2}$ što predstavlja jednačinu kruga sa centrom u koordinatnom početku, poluprečnika $\sqrt{\frac{\pi}{2}}$.



Uvedimo polarne koordinate

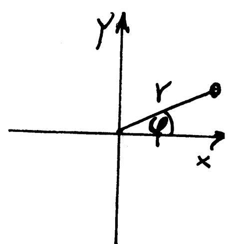
$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$D \xrightarrow{\text{transform.}} D':\begin{cases}0\le r\le\sqrt{\frac{\pi}{2}}\\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$

$x^2+y^2=\ldots=r^2$



$$I=\int_0^{\sqrt{\frac{\pi}{2}}}dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy=\iint_D\cos(x^2+y^2)\,dx\,dy=\iint_{D'}\cos(r^2)\,r\,dr\,d\varphi=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^{\sqrt{\frac{\pi}{2}}}r\cos(r^2)\,dr=\left|\begin{array}{l}r^2=t\\ 2r\,dr=dt\\ r\,dr=\frac{1}{2}dt\\ r\Big|_0^{\sqrt{\frac{\pi}{2}}}\Rightarrow t\Big|_0^{\frac{\pi}{2}}\end{array}\right|=\frac{1}{2}\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi/2}\cos t\,dt=\frac{1}{2}\cdot\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\sin t\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\cdot\pi\cdot 1=\frac{\pi}{2}$$ traženo rješenje

Izračunati dvostruki integral

$$\int_0^{2\pi} d\varphi \int_0^a \rho^2 \sin^2\varphi \, d\rho$$

Rj.

$$\int_0^{2\pi} d\varphi \int_0^a \rho^2 \sin^2\varphi \, d\rho = \int_0^{2\pi} \sin^2\varphi \, d\varphi \int_0^a \rho^2 d\rho = \left| \begin{array}{c} 1 = \sin^2\varphi + \cos^2\varphi \\ \cos 2\varphi = \cos^2\varphi - \sin^2\varphi \\ \hline 1 - \cos 2\varphi = 2\sin^2\varphi \\ \sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi) \end{array} \right|$$

$$= \int_0^{2\pi} \frac{1}{2}(1-\cos 2\varphi) \frac{1}{3}\rho^3\Big|_0^a d\varphi =$$

$$= \frac{a^3}{6}\int_0^{2\pi} (1-\cos 2\varphi)\, d\varphi = \frac{a^3}{6}\left(\varphi\Big|_0^{2\pi} - \frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}\right)$$

$$= \frac{a^3\pi}{3}$$ traženo rješenje

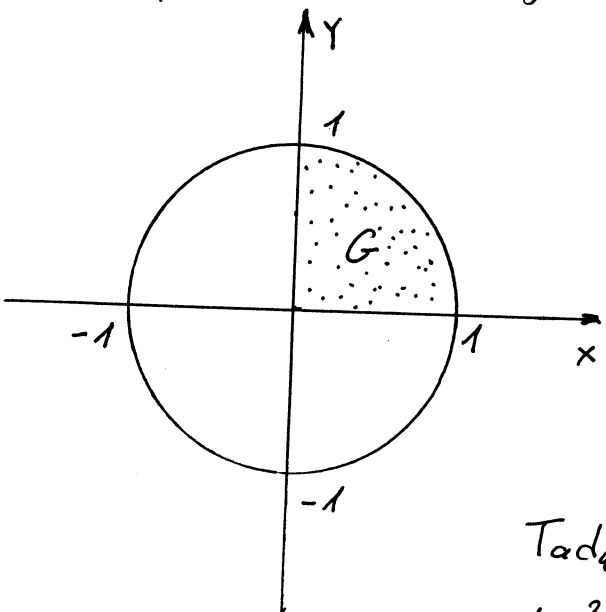
(#) Izračunati dvojni integral

$$I = \iint\limits_G \frac{xy\sqrt{1-x^2-y^2}}{2x^2+y^2}\,dx\,dy$$

gdje je $G = \{(x,y) : x^2+y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$.

Rj.

Skicirajmo oblast integracije G



Ako uvedemo polarne koordinate

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$

$$G \xrightarrow{\text{transformiše}} G' : \begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

Tada

$$1-x^2-y^2 = 1-(x^2+y^2) = 1-\rho^2$$

$$xy = \rho^2\sin\varphi\cos\varphi$$

$$2x^2+y^2 = 2\rho^2\cos^2\varphi + \rho^2\sin^2\varphi =$$

$$= \rho^2(\underbrace{2\cos^2\varphi}_{\cos^2\varphi+\cos^2\varphi} + \sin^2\varphi) = \rho^2(\cos^2\varphi + 1)$$

$$\iint\limits_G \frac{xy\sqrt{1-x^2-y^2}}{2x^2+y^2}\,dx\,dy = \left|\begin{array}{c}\text{uvedimo}\\ \text{polarne}\\ \text{koordinate}\end{array}\right| = \iint\limits_{G'} \frac{\rho^2\sin\varphi\cos\varphi\sqrt{1-\rho^2}}{\rho^2(\cos^2\varphi+1)}\,\rho\,d\rho\,d\varphi$$

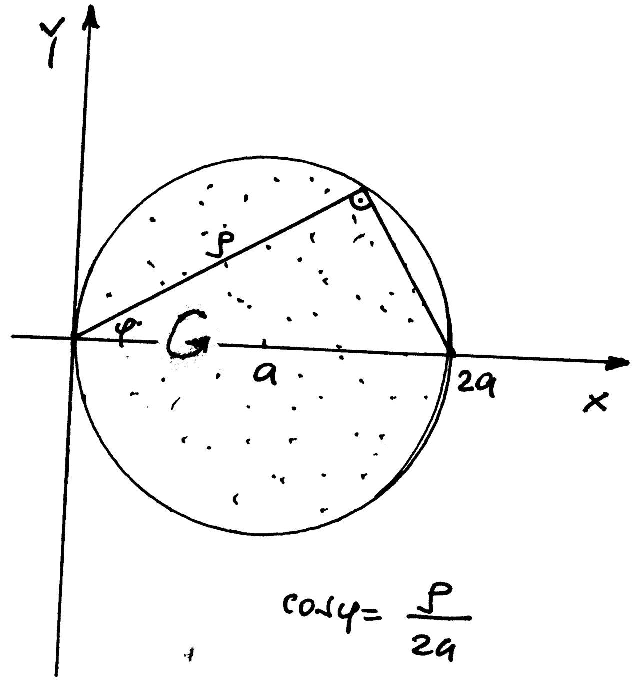
$$= \int_0^{\pi/2} \frac{\sin\varphi\cos\varphi}{\cos^2\varphi+1}\,d\varphi \int_0^1 \rho\sqrt{1-\rho^2}\,d\rho = \left|\begin{array}{l} d(\cos^2\varphi+1) = 2\cos\varphi(-\sin\varphi)\,d\varphi \\ \cos\varphi\sin\varphi\,d\varphi = -\frac{1}{2}d(\cos^2\varphi+1) \\ d(1-\rho^2) = -2\rho\,d\rho \\ \rho\,d\rho = -\frac{1}{2}d(1-\rho^2) \end{array}\right|$$

$$= \int_0^{\pi/2} \frac{-\frac{1}{2}d(\cos^2\varphi+1)}{\cos^2\varphi+1} \int_0^1 \left(-\frac{1}{2}\right)(1-\rho^2)^{\frac{1}{2}}d(1-\rho^2) = \frac{1}{4}\ln|\cos^2\varphi+1|\Big|_0^{\frac{\pi}{2}} \cdot \frac{2}{3}(1-\rho^2)^{\frac{3}{2}}\Big|_0^1 = \frac{1}{6}\ln 2$$

(#) Izračunati $I=\iint\limits_G \left(x+\frac{y^2}{x^2}\right)dx\,dy$ gdje je

$G=\{(x,y):\ x^2+y^2-2ax\le 0,\ a>0\}$.

Rj. Skicirajmo oblast G



$x^2+y^2-2ax=0$

$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

$(x-a)^2+y^2=a^2$

krug sa centrom u tački $C(a,0)$ poluprečnika $r=a$

$\cos\varphi=\frac{\rho}{2a}$

$\rho=2a\cos\varphi$

Uvedimo polarne koordinate

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$G \xrightarrow{\text{transformiše}} G':\ \begin{cases} -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2} \\ 0\le\rho\le 2a\cos\varphi \end{cases}$

$$I=\iint\limits_G\left(x+\frac{y^2}{x^2}\right)dx\,dy=\left|\begin{matrix}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{matrix}\right|=\iint\limits_{G'}\left(\rho\cos\varphi+\frac{\sin^2\varphi}{\cos^2\varphi}\right)\rho\,d\rho\,d\varphi$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\,d\varphi\int_0^{2a\cos\varphi}\rho^2\,d\rho+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\cos^2\varphi}\,d\varphi\int_0^{2a\cos\varphi}\rho\,d\rho=\ldots=\frac{8a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\varphi\,d\varphi+$$

$$+2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\cos^2\varphi}\cdot\cos^2\varphi\,d\varphi=\ldots=a^3\pi+a^2\pi=a^2(a+1)\pi$$ traženo rješenje

\# Izračunati $\iint_D y\,dx\,dy$ gdje je

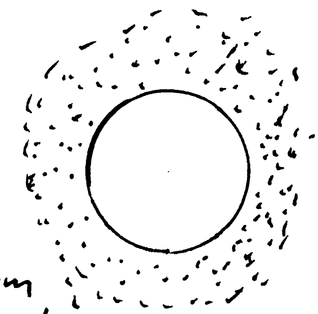
$$D=\{(x,y):\ 1\le x^2+y^2\le 2x,\ y\ge 0\}.$$

Rj:

$1\le x^2+y^2$

$x^2+y^2=1$

krug sa centrom u tački $C(0;0)$ polupr. $r=1$

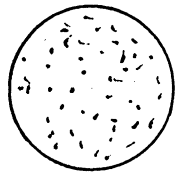


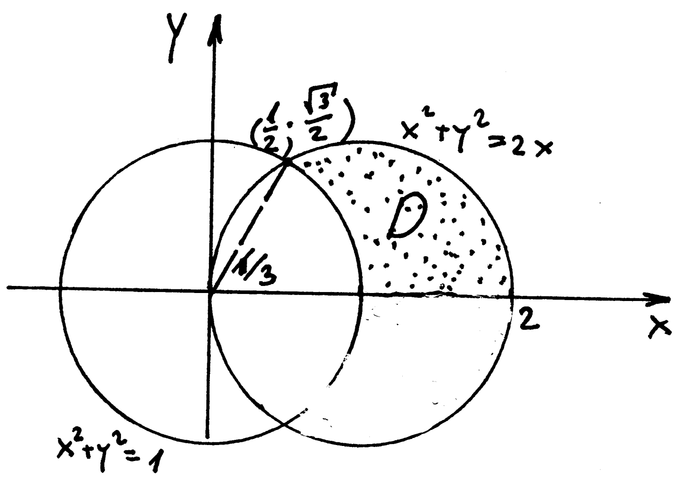
$x^2+y^2\le 2x$

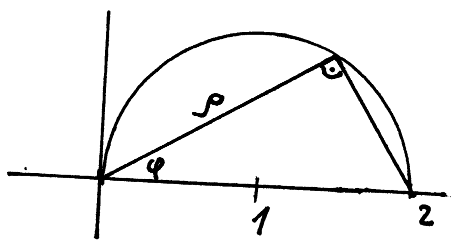
$x^2+y^2=2x$

$x^2-2\cdot x\cdot 1+1+y^2=1$

$(x-1)^2+y^2=1$







$\cos\varphi=\frac{\rho}{2}$

$\rho=2\cos\varphi$

Uvedimo polarne koordinate

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transform.}} D'$

$$D':\begin{cases}0\le\varphi\le\frac{\pi}{3}\\ 1\le\rho\le 2\cos\varphi\end{cases}$$

$$\iint_D y\,dx\,dy=\left|\begin{matrix}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{matrix}\right|=\iint_{D'}\rho\sin\varphi\,\rho\,d\rho\,d\varphi=$$

$$=\int_0^{\frac{\pi}{3}}\sin\varphi\,d\varphi\int_1^{2\cos\varphi}\rho^2\,d\rho=\frac{1}{3}\int_0^{\pi/3}\sin\varphi\underbrace{\rho^3\Big|_1^{2\cos\varphi}}_{8\cos^3\varphi-1}d\varphi=\frac{8}{3}\int_0^{\pi/3}\underbrace{\sin\varphi\cos^3\varphi\,d\varphi}_{-\cos^3\varphi\,d(\cos\varphi)}-\frac{1}{3}\int_0^{\pi/3}\sin\varphi\,d\varphi$$

$$=-\frac{8}{3}\cdot\frac{1}{4}\underbrace{\cos^4\varphi\Big|_0^{\frac{\pi}{3}}}_{\left(\frac{1}{2}\right)^4-1}+\frac{1}{3}\underbrace{\cos\varphi\Big|_0^{\frac{\pi}{3}}}_{\frac{1}{2}-1}=-\frac{2}{3}\cdot\frac{1-16}{16}+\frac{1}{3}\cdot\frac{-1}{2}=\frac{5}{8}-\frac{1}{6}=\frac{11}{24}$$ traženo rješenje

(#) Izračunati $I = \iint\limits_D x\,dx\,dy$ gdje je

$D = \{(x,y) : 1 \le x^2+y^2 \le 4y \ \wedge \ x \le y \ \wedge \ x \ge 0\}.$

Rj.

$1 \le x^2+y^2$

$x^2+y^2=1$

krug sa centrom $C(0,0)$
poluprečnika 1

$x^2+y^2 \le 4y$

$x^2+y^2=4y$

$x^2+y^2-2\cdot y\cdot 2+4=4$

$x^2+(y-2)^2=4$

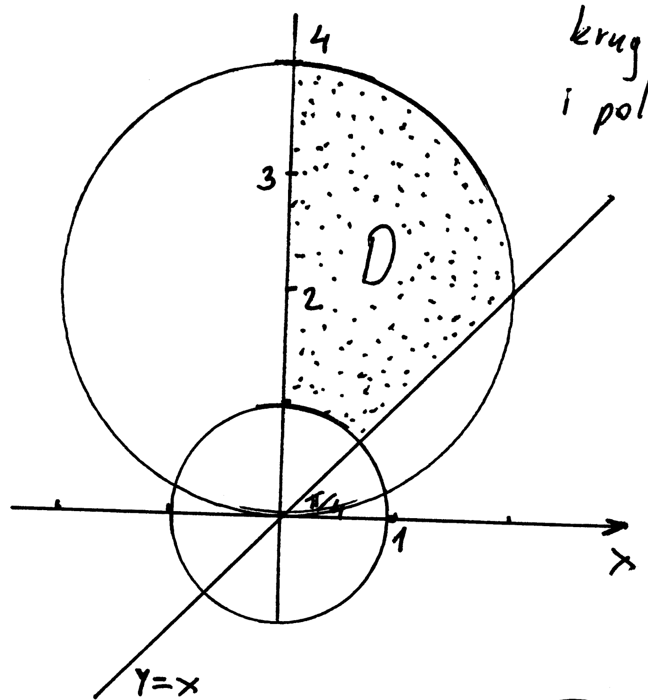
krug sa centrom $C(0;2)$
i poluprečnikom $r=2$

$x \le y$

$x=y$

$x \ge 0$

$x=0$



uvedimo polarne koordinate

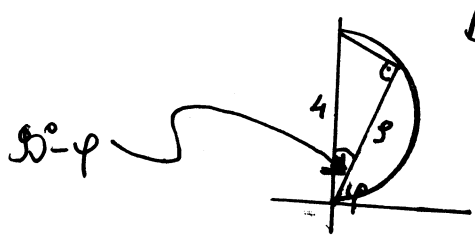
$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transform.}} D'$

$D' : \begin{cases} \frac{\pi}{4} \le \varphi \le \frac{\pi}{2} \\ 1 \le \rho \le 4\sin\varphi \end{cases}$



$\underbrace{\cos(90^\circ-\varphi)}_{=\sin\varphi} = \frac{\rho}{4} \Rightarrow \sin\varphi = \frac{\rho}{4}$

$I = \iint\limits_D x\,dx\,dy = \left| \begin{array}{l} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{array} \right| = \iint\limits_{D'} \rho\cos\varphi\,\rho\,d\rho\,d\varphi = \int\limits_{\pi/4}^{\pi/2} \cos\varphi \int\limits_1^{4\sin\varphi} \rho^2\,d\rho$

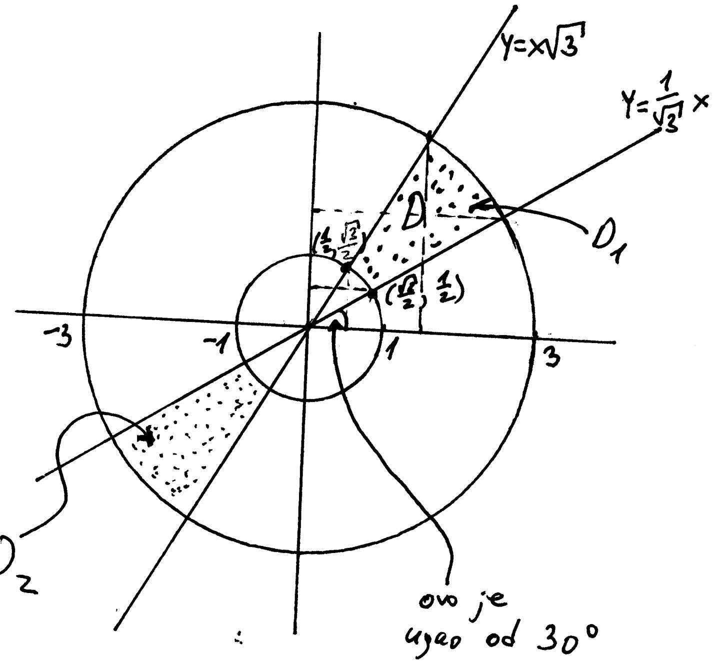
$= \frac{1}{3}\int\limits_{\pi/4}^{\pi/2} \cos\varphi \underbrace{\rho^3\Big|_1^{4\sin\varphi}}_{64\sin^3\varphi-1} d\varphi = \frac{64}{3}\int\limits_{\pi/4}^{\pi/2} \sin^3\varphi\cos\varphi\,d\varphi - \frac{1}{3}\int\limits_{\pi/4}^{\pi/2}\cos\varphi\,d\varphi = \dots = \frac{\sqrt{2}}{6} + \frac{11}{3}$

traženo rješenje

(#) Izračunati dvojni integral $I=\iint_D \operatorname{arctg}\frac{y}{x}\,dx\,dy$, gdje je

$D=\{(x,y):\ 1\le x^2+y^2\le 9,\ \frac{x}{\sqrt{3}}\le y\le x\sqrt{3}\}$.

Rj: Skicirajmo oblast integracije D.



$\frac{x}{\sqrt{3}}=y$ $\qquad y=x\sqrt{3}$

$y=\frac{1}{\sqrt{3}}x$ $\qquad y=x\sqrt{3}$

$$x^2+y^2=1,\quad y=\frac{1}{\sqrt{3}}x$$
$$x^2+\frac{1}{3}x^2=1$$
$$\frac{4}{3}x^2=1$$
$$x^2=\frac{3}{4}$$
$$x_{1,2}=\pm\frac{\sqrt{3}}{2}\Rightarrow y_{1,2}=\pm\frac{1}{2}$$

$$x^2+y^2=9,\quad y=\frac{1}{\sqrt{3}}x$$
$$x^2+\frac{1}{3}x^2=9$$
$$\frac{4}{3}x^2=9$$
$$x^2=\frac{27}{4}$$
$$x_{3,4}=\pm\frac{3\sqrt{3}}{2}$$
$$y_{3,4}=\pm\frac{3}{2}$$

$$x^2+y^2=1,\quad y=x\sqrt{3}$$
$$x^2+3x^2=1$$
$$4x^2=1$$
$$x^2=\frac{1}{4}$$
$$x_{1,2}=\pm\frac{1}{2}\Rightarrow y_{1,2}=\pm\frac{\sqrt{3}}{2}$$

$$x^2+y^2=9,\quad y=x\sqrt{3}$$
$$4x^2=9$$
$$x^2=\frac{9}{4}$$
$$x_{3,4}=\pm\frac{3}{2}\Rightarrow y_{3,4}=\pm\frac{3\sqrt{3}}{2}$$

Na osnovu dobijenih presjeka možemo nacrtati date prave

Ako uvedemo polarne koordinate $x=\rho\cos\varphi$, $y=\rho\sin\varphi$ $\qquad dx\,dy=\rho\,d\rho\,d\varphi$

sa slike vidimo da $D \rightsquigarrow D_1\cup D_2$ gdje $D_1:\begin{cases}1\le\rho\le 3\\ \frac{\pi}{6}\le\varphi\le\frac{\pi}{3}\end{cases}$; $D_2:\begin{cases}1\le\rho\le 3\\ \frac{7\pi}{6}\le\varphi\le\frac{4\pi}{3}\end{cases}$

$$\frac{y}{x}=\frac{\rho\sin\varphi}{\rho\cos\varphi}=\operatorname{tg}\varphi$$

$$I=\iint_D \operatorname{arctg}\frac{y}{x}\,dx\,dy=\left|\begin{array}{l}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{array}\right|=\iint_{D_1\cup D_2}\varphi\rho\,d\rho\,d\varphi=\dots=\frac{1}{4}\cdot 8\cdot\frac{\pi^2}{12}+\frac{1}{4}\cdot 8\,\frac{5\pi^2}{12}=\frac{\pi^2}{6}+\frac{5\pi^2}{6}=\pi^2$$

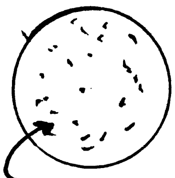
(#) Izračunati $\iint_D y\,dx\,dy$ gdje je $D=\{(x,y): x^2+y^2\leq 1, x^2+y^2\leq 2x, y\geq 0\}$

Rj.

$D=\{(x,y): x^2+y^2\leq 1, \quad x^2+y^2\leq 2x, \quad y\geq 0\}$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

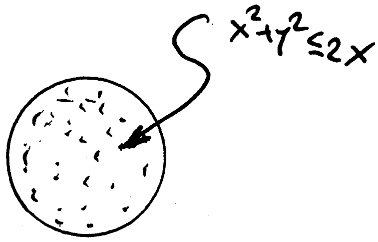
$1=x^2+y^2$ krug sa centrom u $C(0;0)$ polupr. $r=1$



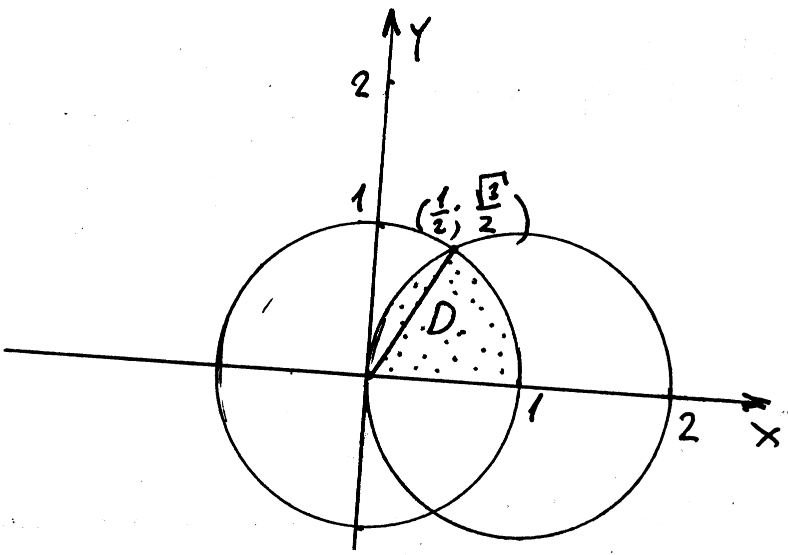
$x^2+y^2=2x$

$x^2-2\cdot x+1+y^2=1$

$(x-1)^2+y^2=1$ krug sa centrom u $C(1;0)$ polupr. $r=1$



Skicirajmo oblast D



Ako uvedemo polarne koordinate

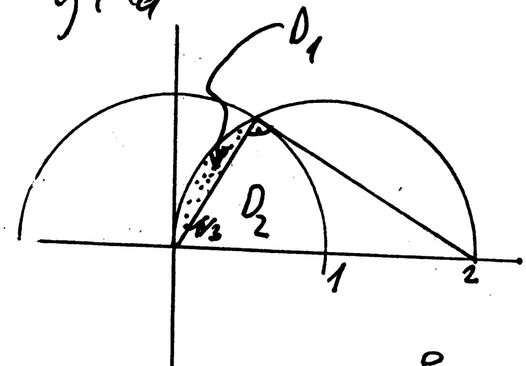
$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transf.}} D'$

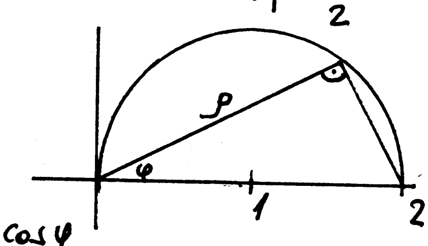
oblast D' možemo podjeliti na dva dijela



$D_2: \begin{cases} 0\leq\rho\leq 1 \\ 0\leq\varphi\leq\pi/3 \end{cases}$

$D_1: \begin{cases} \frac{\pi}{3}\leq\varphi\leq\frac{\pi}{2} \\ 0\leq\rho\leq 2\cos\varphi \end{cases}$

$\cos\varphi=\frac{\rho}{2}$



$\rho=2\cos\varphi$

$$\iint_D y\,dx\,dy = \left| \begin{matrix} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{matrix} \right| = \iint_{D'} \rho \sin\varphi \, \rho \, d\rho \, d\varphi =$$

$$= \iint_{D_1 \cup D_2} \rho^2 \sin\varphi \, d\rho \, d\varphi = \iint_{D_1} \rho^2 \sin\varphi \, d\rho \, d\varphi + \iint_{D_2} \rho^2 \sin\varphi \, d\rho \, d\varphi$$

$$\iint_{D_1} \rho^2 \sin\varphi \, d\rho \, d\varphi = \int_{\pi/3}^{\pi/2} \sin\varphi \, d\varphi \int_0^{2\cos\varphi} \rho^2 \, d\rho = \frac{1}{3} \int_{\pi/3}^{\pi/2} \sin\varphi \, \rho^3 \Big|_0^{2\cos\varphi} d\varphi$$

$$= \frac{8}{3} \int_{\pi/3}^{\pi/2} \cos^3\varphi \sin\varphi \, d\varphi = -\frac{8}{3} \int_{\pi/3}^{\pi/2} \cos^3\varphi \, d(\cos\varphi) = -\frac{8}{3} \cdot \frac{1}{4} \cos^4\varphi \Big|_{\pi/3}^{\pi/2} = -\frac{2}{3}\left(0 - \left(\frac{1}{2}\right)^4\right)$$

$$= \frac{2}{3} \cdot \frac{1}{16} = \frac{1}{24}$$

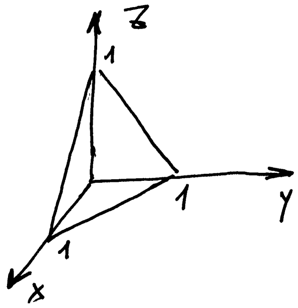
$$\iint_{D_2} \rho^2 \sin\varphi \, d\rho \, d\varphi = \int_0^{\pi/3} \sin\varphi \int_0^1 \rho^2 \, d\rho = \frac{1}{3} \rho^3 \Big|_0^1 \cdot \underbrace{(-\cos\varphi) \Big|_0^{\pi/3}}_{-\left(\frac{1}{2} - 1\right)} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

$$\iint_D y\,dx\,dy = \frac{1}{24} + \frac{1 \cdot 4}{6 \cdot 4} = \frac{5}{24} \quad \text{traženo rješenje}$$

\# Izračunati trostruki integral $I = \iiint\limits_{\Omega} \frac{dx\,dy\,dz}{(x+y+z+1)^3}$,

ako je Ω oblast ograničena koordinatnim ravnima i ravni $x+y+z=1$.

Rj. $x+y+z=1$ je ravan koja na koordinatnim osama prolazi kroz tačke $(1,0,0)$, $(0,1,0)$ i $(0,0,1)$



$$\Omega : \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 0 \le z \le 1-x-y \end{cases}$$

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3} \overset{(*)}{=}$$

$$\int \frac{dz}{(x+y+z+1)^2} = \left| \begin{matrix} x+y+z+1 = t \\ dz = dt \end{matrix} \right| = \int \frac{dt}{t^3} = \int t^{-3} dt = \frac{t^{-2}}{-2} + C =$$

$$= \frac{-1}{2(x+y+z+1)^2} + C$$

$$\overset{(*)}{=} \int_0^1 dx \int_0^{1-x} \frac{-1}{2(x+y+z+1)^2} \Bigg|_0^{1-x-y} dy = \int_0^1 dx \int_0^{1-x} \Bigg(\frac{-1}{2(x+y+1-x-y+1)^2} -$$

$$- \frac{-1}{2(x+y+0+1)^2} \Bigg) dy = \int_0^1 dx \int_0^{1-x} \left(-\frac{1}{8} + \frac{1}{2(x+y+1)^2} \right) dy =$$

$$= -\frac{1}{2} \int_0^1 dx \int_0^{1-x} \left(\frac{1}{4} - \frac{1}{(x+y+1)^2} \right) dy \overset{(**)}{=} -\frac{1}{2} \int_0^1 \left(\frac{1}{4} y \Big|_0^{1-x} + \frac{1}{x+y+1} \Big|_0^{1-x} \right) dx$$

$$\int \frac{dy}{(x+y+1)^2} = \left| \begin{matrix} x+y+1 = t \\ dy = dt \end{matrix} \right| = \int \frac{dt}{t^2} = \frac{t^{-1}}{-1} + C = \frac{-1}{t} + C = \frac{-1}{x+y+1} + C \quad \dots (**)$$

$$= -\frac{1}{2} \int_0^1 \left(\frac{1}{4}(1-x) + \frac{1}{2} - \frac{1}{x+1} \right) dx = -\frac{1}{2} \Bigg(\overbrace{\frac{1}{4} x \Big|_0^1 - \frac{1}{4} \cdot \frac{x^2}{2} \Big|_0^1}^{\frac{1}{8}} + \frac{1}{2} x \Big|_0^1 - \underbrace{\ln|x+1| \Big|_0^1}_{\ln 2 - \ln 1} \Bigg) = \frac{1}{2} \ln 2 - \frac{5}{16}$$

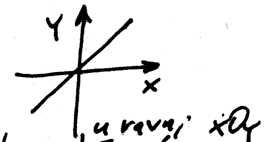
(#) Izračunati trostruki integral $I=\iiint\limits_{\Omega} z\,dx\,dy\,dz$, ako je

$\Omega: y=x,\ y=2x,\ 2x=1,\ x^2+y^2+z^2=1,\ z\geq 0$

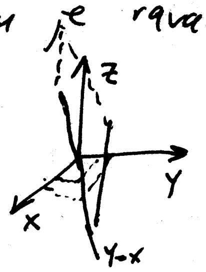
(oblast Ω je ograničena ovim površinama).

Rj: Komentarišimo površi koje čine Ω.

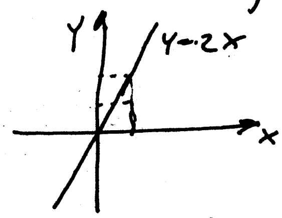
$y=x$ u ravni je prava



$y=x$ u prostoru je ravan koja u ravni xOy sadrži pravu $y=x$

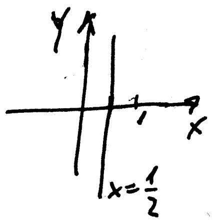


$y=2x$ u ravni je prava

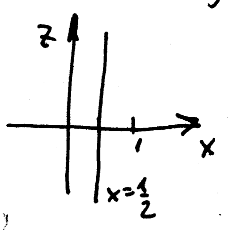


$y=2x$ u prostoru je ravan koja u ravni xOy sadrži pravu $y=2x$

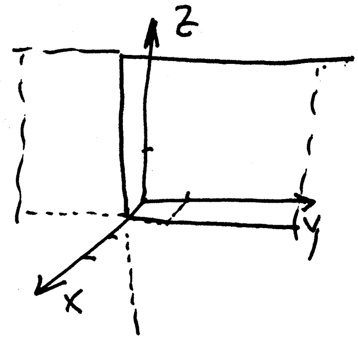
$2x=1$ u ravni xOy je prava



u ravni xOz je isto prava



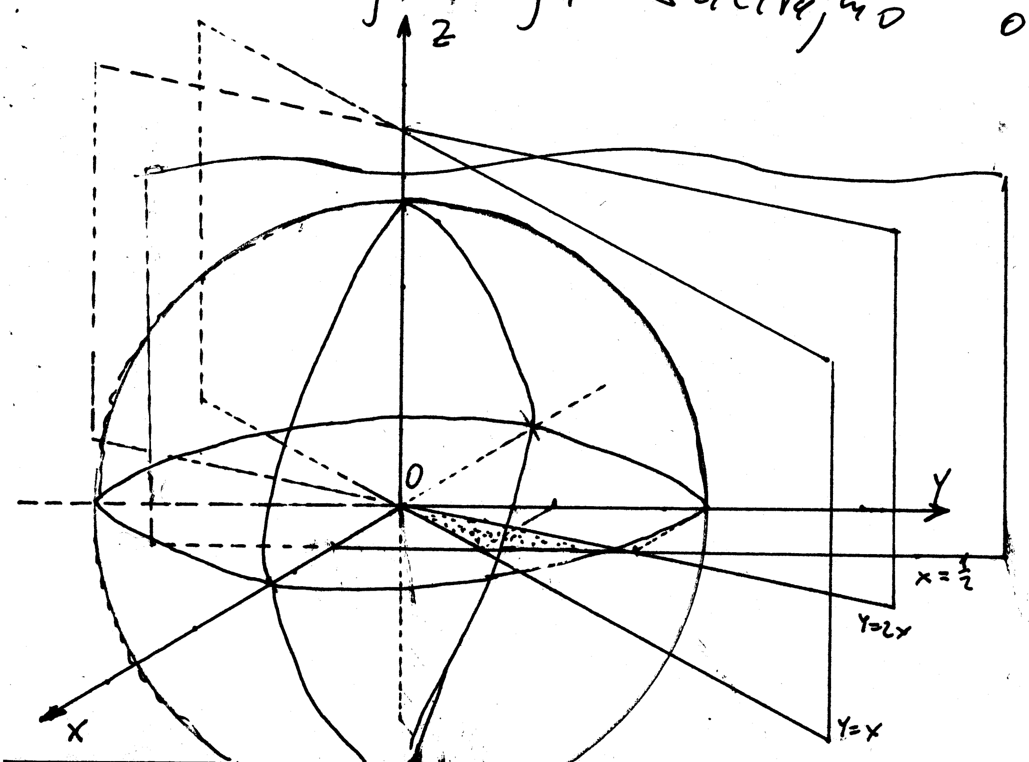
U prostoru to je ravan koja sadrži u xOz ravni pravu $x=\frac{1}{2}$ i u xOy ravni pravu $x=\frac{1}{2}$



$x=\frac{1}{2}$ je ravan koja je paralelna sa yOz osom

$x^2+y^2+z^2=1$ je jednačina kružnice

Na osnovu svega ovoga skicirajmo oblast Ω.



Oblast Ω je kružni isječak čija projekcija na xOy ravan je predstavljena tačkama na slici. Možemo zaključiti

$$\Omega: \begin{cases} 0 \leq x \leq \frac{1}{2} \\ x \leq y \leq 2x \\ 0 \leq z \leq \sqrt{1-x^2-y^2} \end{cases}$$

$$I=\iiint\limits_{\Omega} z\,dx\,dy\,dz = \int_0^{\frac12} dx \int_x^{2x} dy \int_0^{\sqrt{1-x^2-y^2}} z\,dz = \int_0^{\frac12} dx \int_x^{2x} \frac12 z^2 \Big|_0^{\sqrt{1-x^2-y^2}} dy =$$

$$= \frac12 \int_0^{\frac12} dx \int_x^{2x} (1-x^2-y^2)\,dy = \frac12 \int_0^{\frac12} \left(y\Big|_x^{2x} - x^2 y\Big|_x^{2x} - \frac13 y^3\Big|_x^{2x} \right) dx =$$

$$= \frac12 \int_0^{1/2} \left(x - x^3 - \frac13 7x^3 \right) dx = \frac12 \int_0^{1/2} \left(x - \frac{10}{3} x^3 \right) dx = \frac12 \left(\frac12 x^2 \Big|_0^{\frac12} - \frac{\cancel{10}^{\,5}}{3} \cdot \frac{1}{\cancel{4}_{\,2}} x^4 \Big|_0^{\frac12} \right)$$

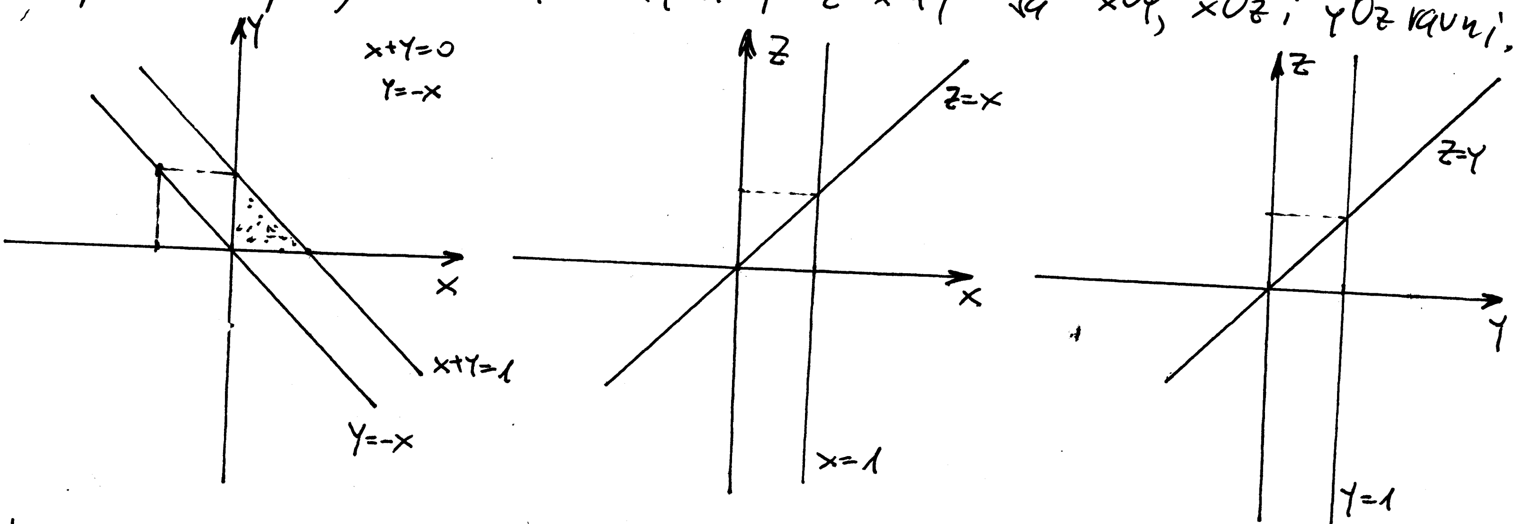
$$= \frac12 \left(\frac12 \cdot \frac14 - \frac56 \cdot \frac{1}{16} \right) = \frac12 \left(\frac18 - \frac{5}{} \right) = \frac12 \cdot \frac{12-5}{96} = \frac{7}{192}$$

(#) Izračunati trojni integral

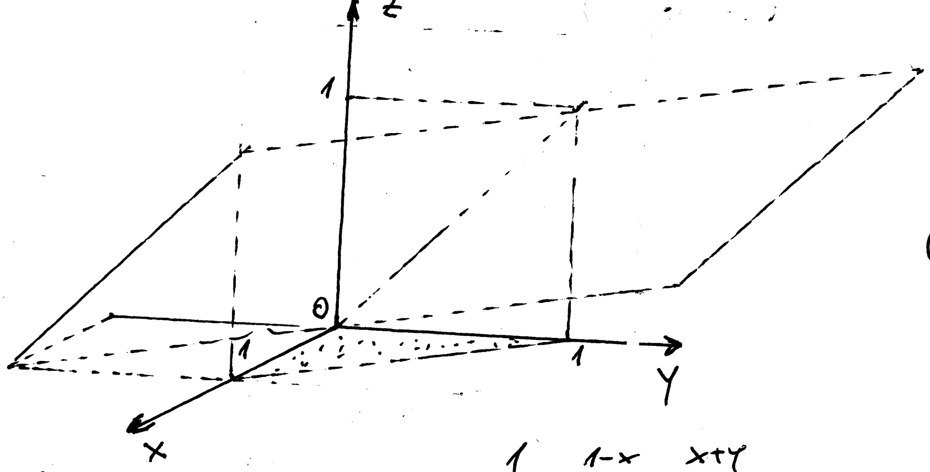
$$I=\iiint\limits_G \frac{1}{(1+z)^3}\,dx\,dy\,dz$$

gde je oblast G u I oktantu ograničena ravnima $x+y=1$, $z=x+y$, $x=0$, $y=0$, $z=0$.

Rj. Napravimo presjek ravni $x+y=1$ i $z=x+y$ sa xOy, xOz i yOz ravni.



Iz presjeka vidimo da je ravan $x+y=1$ paralelna sa z osom a da je oblast G odozgo ograničena sa $z=x+y$ ravni



$$G:\begin{cases} 0\le x\le 1\\ 0\le y\le 1-x\\ 0\le z\le x+y\end{cases}$$

$$I=\iiint\limits_G \frac{1}{(1+z)^2}\,dx\,dy\,dz=\int_0^1 dx\int_0^{1-x} dy\int_0^{x+y}(1+z)^{-3}\,d(1+z)=\frac{1}{-2}\int_0^1 dx\int_0^{1-x}\overbrace{(1+z)^{-2}\Big|_0^{x+y}}^{(1+(x+y))^{-2}-1^{-2}}dy=$$

$$=-\frac{1}{2}\int_0^1 dx\int_0^{1-x}(1+x+y)^{-2}\,d(1+x+y)-\left(-\frac{1}{2}\right)\int_0^1 dx\int_0^{1-x}dy=-\frac{1}{2}\int_0^1(-1)(1+x+y)^{-1}\Big|_0^{1-x}dx+$$

$$+\frac{1}{2}\int_0^1 (1-x)\,dx = \frac{1}{2}\int_0^1 \left(2^{-1}-(1+x)^{-1}\right)dx + \frac{1}{2}\int_0^1 (1-x)\,dx =$$

$$\frac{1}{2} \quad -(1+x)^{-1}$$

$$= \frac{1}{2}\left(\frac{1}{2}x\Big|_0^1 - \ln(1+x)\Big|_0^1 + x\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1 \right) =$$

$$= \frac{1}{2}\left(\frac{1}{2} - \ln 2 + 1 - \frac{1}{2}\right) = \frac{1}{2}(1-\ln 2)$$

traženo rješenje

(#) Izračunati trostruki integral $J=\iiint\limits_W (x^2+y^2+z^2)\,dx\,dy\,dz$

gdje je oblast W ograničena površinom $3(x^2+y^2)+z^2=3a^2$.

Rj. Skicirajmo oblast W

$3(x^2+y^2)+z^2=3a^2$

$3x^2+3y^2+z^2=3a^2 \quad /:3a^2$

$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{3a^2}=1$

jednačina elipse

za elipsu $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ uopštene sferne koordinate glase $x=ar\sin\varphi\cos\theta$

$y=br\sin\varphi\sin\theta$

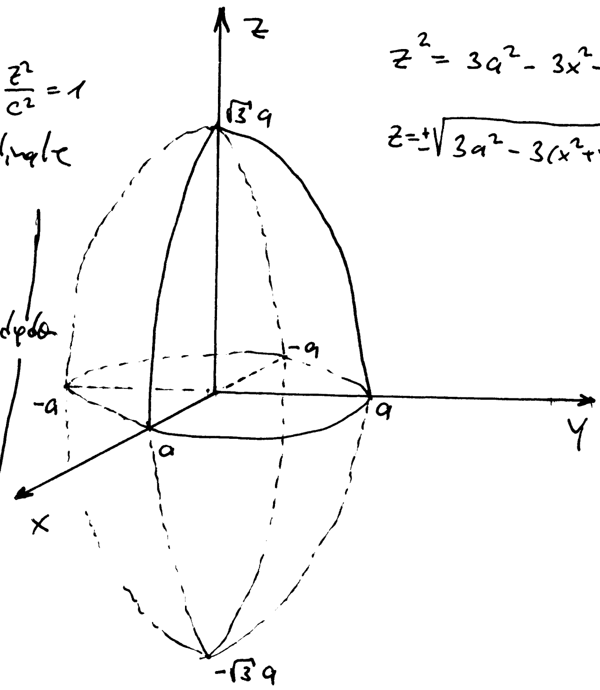
$z=cr\cos\varphi$

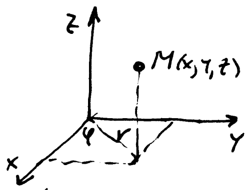
$dxdydz=abcr^2\sin\varphi\,dr\,d\varphi\,d\theta$

U ovom slučaju uopštene sferne koordinate ne mogu na lagan način riješit zadatak

$z^2=3a^2-3x^2-3y^2$

$z=\pm\sqrt{3a^2-3(x^2+y^2)}$





Uvodimo cilindrične koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$dxdydz=r\,dr\,d\varphi\,dz$

$x^2+y^2+z^2=r^2+z^2$

$W \xrightarrow{\text{transformije}} W' : \begin{cases} 0\le r\le a \\ 0\le\varphi\le 2\pi \\ -\sqrt{3(a^2-(x^2+y^2))}\le z\le\sqrt{3(a^2-(x^2+y^2))} \\ -\sqrt{3(a^2-r^2)}\le z\le\sqrt{3(a^2-r^2)} \end{cases}$

$J=\iiint\limits_W (x^2+y^2+z^2)\,dx\,dy\,dz = \left|\text{uvodimo cilindrične koordinate}\right| = \iiint\limits_{W'} (r^2+z^2)\,r\,dr\,d\varphi\,dz =$

$= \int_0^{2\pi} d\varphi \int_0^a dr \int_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}} (r^2+z^2)\,r\,dz = \int_0^{2\pi} d\varphi \int_0^a \left(r^2 z\Big|_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}} + \frac{z^3}{3}\Big|_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}} \right) r\,dr$

$= \int_0^{2\pi} d\varphi \int_0^a \left(r^2\, 2\sqrt{3}\sqrt{a^2-r^2} + \frac{1}{3}\left(3\sqrt{3}\sqrt{(a^2-r^2)^3} + 3\sqrt{3}\sqrt{(a^2-r^2)^3}\right)\right) r\,dr =$

(ako ovo izvučemo ispred zagrade)

$= \int_0^{2\pi} d\varphi \int_0^a \left(2\sqrt{3}\,r^2\underline{\underline{\sqrt{a^2-r^2}}} + 2\sqrt{3}\,(a^2-r^2)\underline{\underline{\sqrt{a^2-r^2}}}\right) r\,dr = 2\sqrt{3}\,a^2 \int_0^{2\pi} d\varphi \int_0^a \sqrt{a^2-r^2}\,r\,dr$

$= \left| d(a^2-r^2) = -2r\,dr \right| = -\sqrt{3}\,a^2 \int_0^{2\pi} d\varphi \int_0^a \sqrt{a^2-r^2}\,d(a^2-r^2) = \ \ldots\ = \frac{1}{\sqrt{3}}\,4\pi a^5$ rješenje

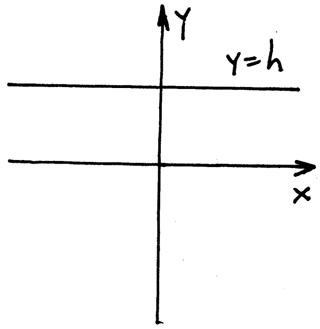
(#) Izračunati trostruki integral

$$K = \iiint_T y \, dx \, dy \, dz$$

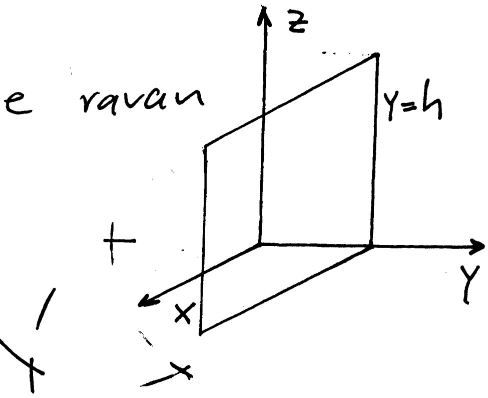
gdje je oblast T ograničena površinama $y = \sqrt{x^2+z^2}$ i $y = h$, $h > 0$.

Rj. Pokušajmo skicirati oblast T.

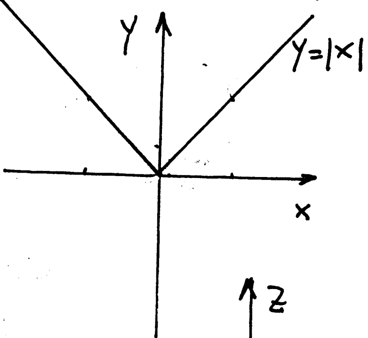
U xOy-ravni $y=h$ je prava.



U prostoru $y=h$ je ravan

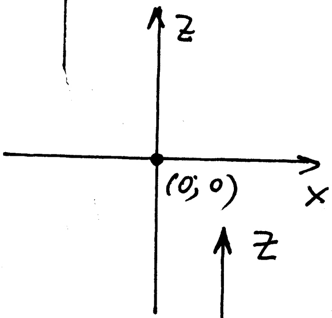


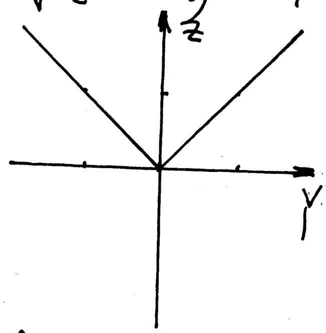
U xOy-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{x^2}$



U xOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $0=\sqrt{x^2+z^2}$ tj. tačka $(0;0)$.

U yOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{z^2}$ tj. $y=|z|$.

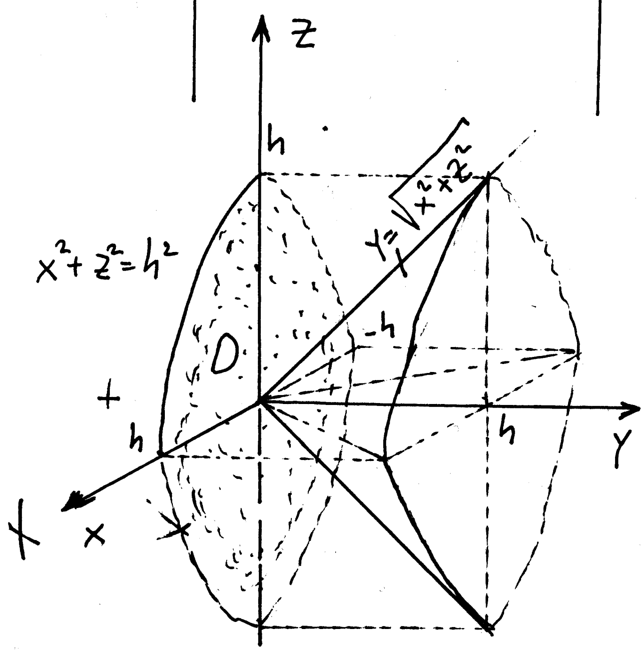




Ako napravimo presjek površina $y=\sqrt{x^2+z^2}$ i $y=h$ dobićemo $h=\sqrt{x^2+z^2}$ tj.

$$x^2+z^2=h^2$$

(krug poluprečnika h)



Oblast T (pola čunja) je prikazan na slici lijevo.

Ako napravimo projekciju oblasti T na xOz ravan dobićemo sljedeće granice:

$$T: \begin{cases} -h \le x \le h \\ -\sqrt{h^2-z^2} \le z \le \sqrt{h^2-z^2} \\ \sqrt{x^2+z^2} \le y \le h \end{cases}$$

Pomoću pravougaonih koordinata dati trostruki integral je teško izračunati.

Uvodimo cilindrične koordinate i to

$$x = r\cos\varphi$$
$$z = r\sin\varphi$$
$$y = y$$
$$dx\,dy\,dz = r\,dr\,d\varphi\,dy$$

$$T \xrightarrow{\text{transformiše}} T': \begin{cases} 0 \le r \le h \\ 0 \le \varphi \le 2\pi \\ r \le y \le h \end{cases}$$

Prema tome

$$K = \iiint_T y\,dx\,dy\,dz = \left| \begin{array}{c} \text{uvodimo} \\ \text{cilindrične} \\ \text{koordinate} \end{array} \right| = \iiint_{T'} y\,r\,dr\,d\varphi\,dy =$$

$$= \int_0^{2\pi} d\varphi \int_0^h r\,dr \int_r^h y\,dy = \int_0^{2\pi} d\varphi \int_0^h r \underbrace{\frac{1}{2}y^2\Big|_r^h}_{h^2-r^2} dr =$$

$$= \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^h (rh^2 - r^3)\,dr = \frac{1}{2}\int_0^{2\pi} \Big(\underbrace{\frac{1}{2}r^2h^2\Big|_0^h}_{\frac{1}{2}r^4} \underbrace{-\frac{1}{4}r^4\Big|_0^h}_{-\frac{1}{4}h^4} \Big) d\varphi$$

$$= \frac{1}{2}\cdot\frac{1}{4}h^4 \int_0^{2\pi} d\varphi = \frac{1}{8}h^4\,\varphi\Big|_0^{2\pi} = \frac{h^4\pi}{4}$$

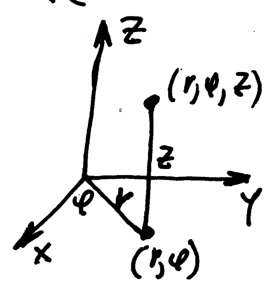
traženo rješenje

(#) Dati trojni integral $\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz$ transformisati na trostruki u cilindričnim koordinatama (sa određenim posebnim granicama integracije) ako je Ω oblast u prvom oktantu ograničen cilindrom $x^2+y^2=R^2$ i ravnima $z=0$, $z=1$, $y=x$ i $y=x\sqrt{3}$

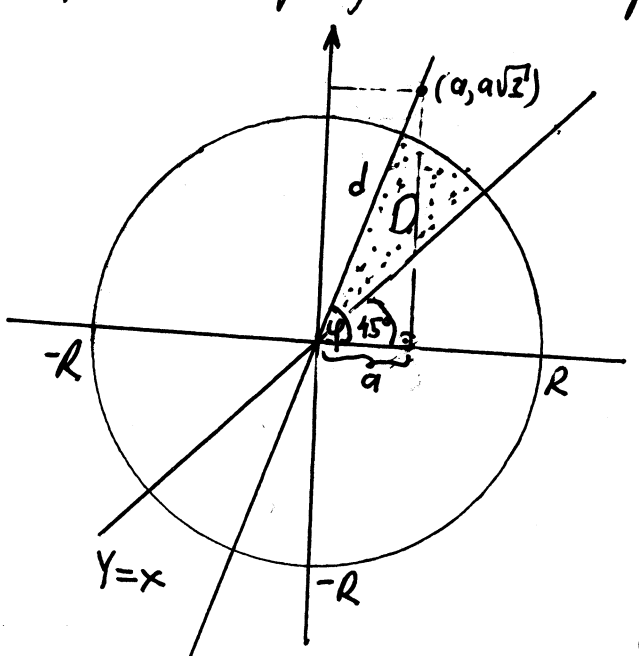
Rj.

Cilindrične koordinate glase

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$
$$dx\,dy\,dz = r\,dr\,d\varphi\,dz$$



Napravimo presjek datih površina sa xOy ravni.



$$\cos\varphi = \frac{a}{d} = \frac{a}{2a} = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{3}$$
$$d^2 = a^2 + 3a^2 = 4a^2$$
$$d = 2a$$

Sad nije teško vidjeti da je

$$\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int_0^1 dz \int_{\pi/4}^{\pi/3} d\varphi \int_0^R f(r\cos\varphi, r\sin\varphi, z)\, r\,dr.$$

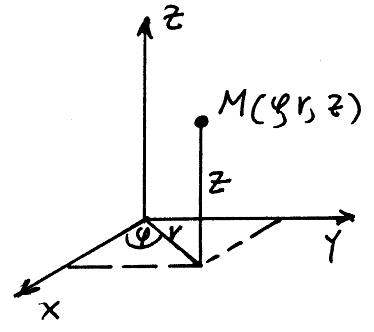
$\Omega \xrightarrow{\text{transformiše}} \Omega'$

$$\Omega' : \begin{cases} 0 \le r \le R \\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{3} \\ 0 \le z \le 1 \end{cases}$$

Dat je trostruki integral $\int_0^{2\pi} d\varphi \int_0^2 r^3 dr \int_0^{\sqrt{4-r^2}} dz$ u

cilindričnim koordinatama. Skicirati oblast integracije i izračunati taj integral prelazeći na sferne koordinate.

Rj: U cilindričnim koordinatama proizvoljna tačka M je opisana na slijedeći način



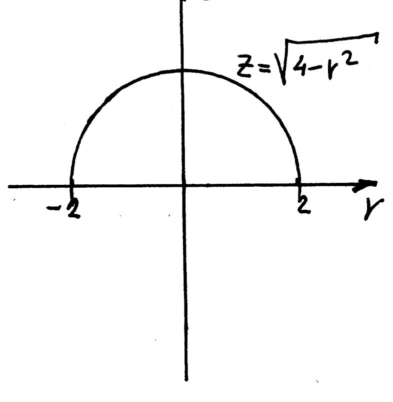
$$\Omega: \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 2 \\ 0 \le z \sqrt{4-r^2} \end{cases}$$

Na osnovu izgleda oblasti Ω vidimo da je projekcija figure na xOy ravan oblika

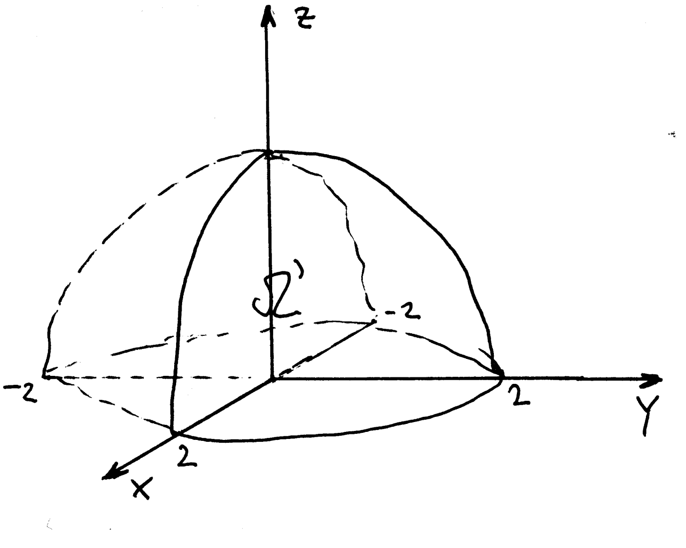
$$D: \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 2 \end{cases}$$

tj. krug sa centrom u koordinatnom početku poluprečnika 2.

Ako za fiksirano φ posmatramo rOz ravan imamo



Prema tome oblast integracije Ω je polulopta



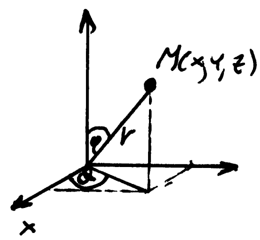
Cilindrične koordinate glase

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$
$dx\,dy = r\,dr\,d\varphi$

Tako da bi prelaskom na pravougaone koordinate sad imali

$$\int_0^{2\pi} d\varphi \int_0^{2} r^3 dr \int_0^{\sqrt{1-r^2}} dz = \iiint_{\Omega} r^2\, r\, dr\, d\varphi\, dz = \left| \begin{array}{l} \text{prelazimo na pravougaone} \\ \text{koordinate} \\ \Omega \xrightarrow{\text{transformiše}} \Omega' \\ r\,dr\,d\varphi = dx\,dy \\ r^2 = r^2(\sin^2\varphi + \cos^2\varphi) \\ \quad = r^2\sin^2\varphi + r^2\cos^2\varphi \\ \quad = (r\sin\varphi)^2 + (r\cos\varphi)^2 \\ \quad = x^2+y^2 \end{array} \right| =$$

$$= \iiint_{\Omega'} (x^2+y^2)\, dx\, dy\, dz$$



Sferne koordinate glase

$$x = r\sin\varphi\cos\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\varphi$$
$$dx\,dy\,dz = r^2\sin\varphi\, dr\, d\varphi\, d\theta$$

$$\Omega' \xrightarrow{\text{transformiše}} \Omega'': \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \pi/2 \\ 0 \le \theta \le 2\pi \end{cases} \qquad x^2+y^2 = r^2\sin^2\varphi$$

$$\iiint_{\Omega'} (x^2+y^2)\, dx\, dy\, dz = \left| \begin{array}{l} \text{uvodimo} \\ \text{sferne} \\ \text{koordinate} \end{array} \right| = \iiint_{\Omega''} r^2\sin^2\varphi\; r^2\sin\varphi\, dr\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi} d\theta \int_0^{2} r^4 dr \int_0^{\pi/2} \sin^3\varphi\, d\varphi \overset{(*)}{=} \theta\Big|_0^{2\pi} \cdot \frac{1}{5} r^5 \Big|_0^{2} \cdot \frac{2}{3} = \frac{2^7}{15}\pi$$

← traženo rješenje

$$\int_0^{\pi/2} \sin^3\varphi\, d\varphi = \int_0^{\pi/2} \sin\varphi \underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi}\, d\varphi = \left| \begin{array}{l} d(\sin\varphi) = \cos\varphi\, d\varphi \\ d(\cos\varphi) = -\sin\varphi\, d\varphi \end{array} \right| = -\int_0^{\pi/2} (1-\cos^2\varphi)\, d(\cos\varphi)$$

$$= -\left(\cos\varphi\Big|_0^{\pi/2} - \frac{1}{3}\cos^3\varphi\Big|_0^{\pi/2} \right) = -\left((0-1) - \frac{1}{3}(0-1) \right) = -\left(-1+\frac{1}{3}\right) = \frac{2}{3} \quad \ldots (*)$$

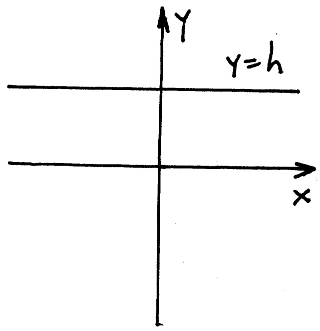
(#) Izračunati trostruki integral

$$K=\iiint\limits_{T} y\,dx\,dy\,dz$$

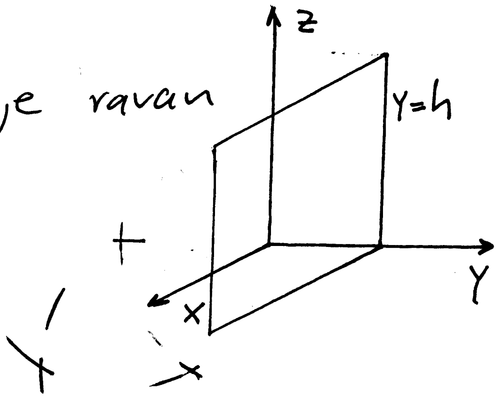
gdje je oblast T ograničena površinama $y=\sqrt{x^2+z^2}$ i $y=h$, $h>0$.

Rj. Pokušajmo skicirati oblast T.

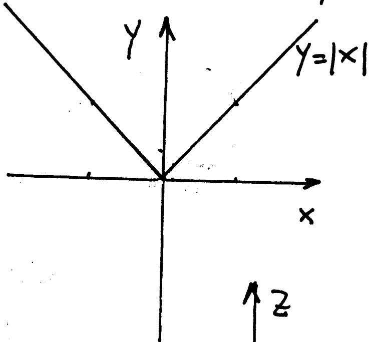
U xOy-ravni $y=h$ je prava.



U prostoru $y=h$ je ravan

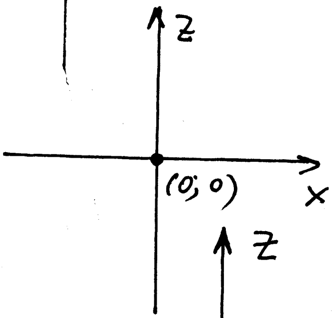


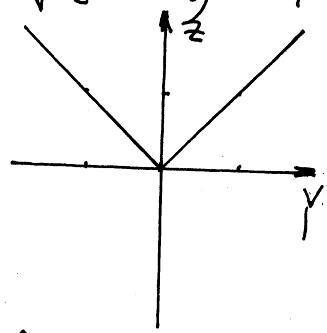
U xOy-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{x^2}$



U xOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $0=\sqrt{x^2+z^2}$ tj. tačka $(0;0)$.

U yOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{z^2}$ tj. $y=|z|$.

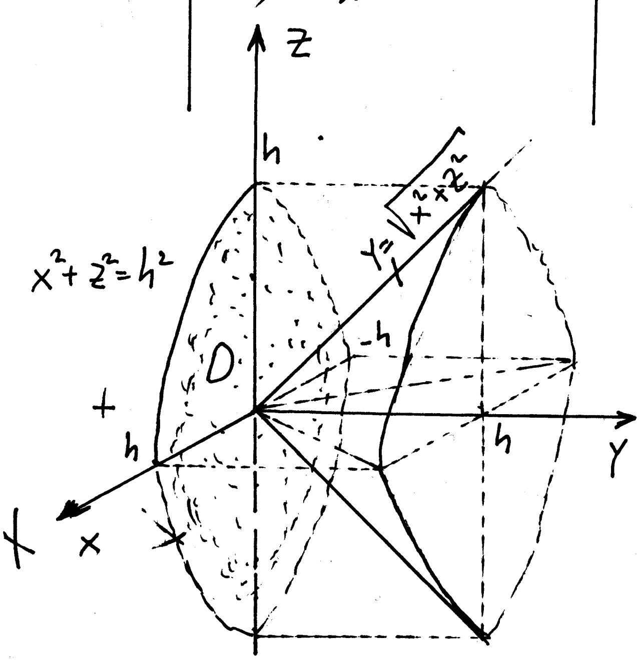




Ako napravimo presjek površina $y=\sqrt{x^2+z^2}$ i $y=h$ dobićemo $h=\sqrt{x^2+z^2}$ tj.

$$x^2+z^2=h^2$$

(krug poluprečnika h)



Oblast T (pola čunja) je prikazan na slici lijevo. Ako napravimo projekciju oblasti T na xOz ravan dobićemo sljedeće granice:

$$T: \begin{cases} -h \le x \le h \\ -\sqrt{h^2-z^2} \le z \le \sqrt{h^2-z^2} \\ \sqrt{x^2+z^2} \le y \le h \end{cases}$$

Pomoću pravougaonih koordinata dati trostruki integral je teško izračunati.

Uvodimo cilindrične koordinate i to

$$x = r\cos\varphi$$
$$z = r\sin\varphi$$
$$y = y$$
$$dx\,dy\,dz = r\,dr\,d\varphi\,dy$$

$$T \xrightarrow{\text{transformište}} T': \begin{cases} 0 \le r \le h \\ 0 \le \varphi \le 2\pi \\ r \le y \le h \end{cases}$$

Prema tome

$$K = \iiint_T y\,dx\,dy\,dz = \left| \begin{matrix} \text{uvodimo} \\ \text{cilindrične} \\ \text{koordinate} \end{matrix} \right| = \iiint_{T'} y\,r\,dr\,d\varphi\,dy =$$

$$= \int_0^{2\pi} d\varphi \int_0^h r\,dr \int_r^h y\,dy = \int_0^{2\pi} d\varphi \int_0^h r \underbrace{\frac{1}{2}y^2\Big|_r^h}_{h^2-r^2} dr =$$

$$= \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^h (rh^2 - r^3)\,dr = \frac{1}{2}\int_0^{2\pi} \Big(\underbrace{\frac{1}{2}r^2h^2\Big|_0^h}_{\frac{1}{2}r^4} \underbrace{-\frac{1}{4}r^4\Big|_0^h}_{-\frac{1}{4}h^4} \Big)\,d\varphi$$

$$= \frac{1}{2}\cdot\frac{1}{4}h^4 \int_0^{2\pi} d\varphi = \frac{1}{8}h^4\,\varphi\Big|_0^{2\pi} = \frac{h^4\pi}{4}$$

traženo rješenje

Izračunati integral

$$\iiint_{\Omega} \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz$$

gdje je $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le z,\ x^2+y^2 \le z^2\}$.

Rj.

$x^2+y^2+z^2 = z$ je jednačina sfere 

$x^2+y^2 = z^2$ je jednačina čunja 

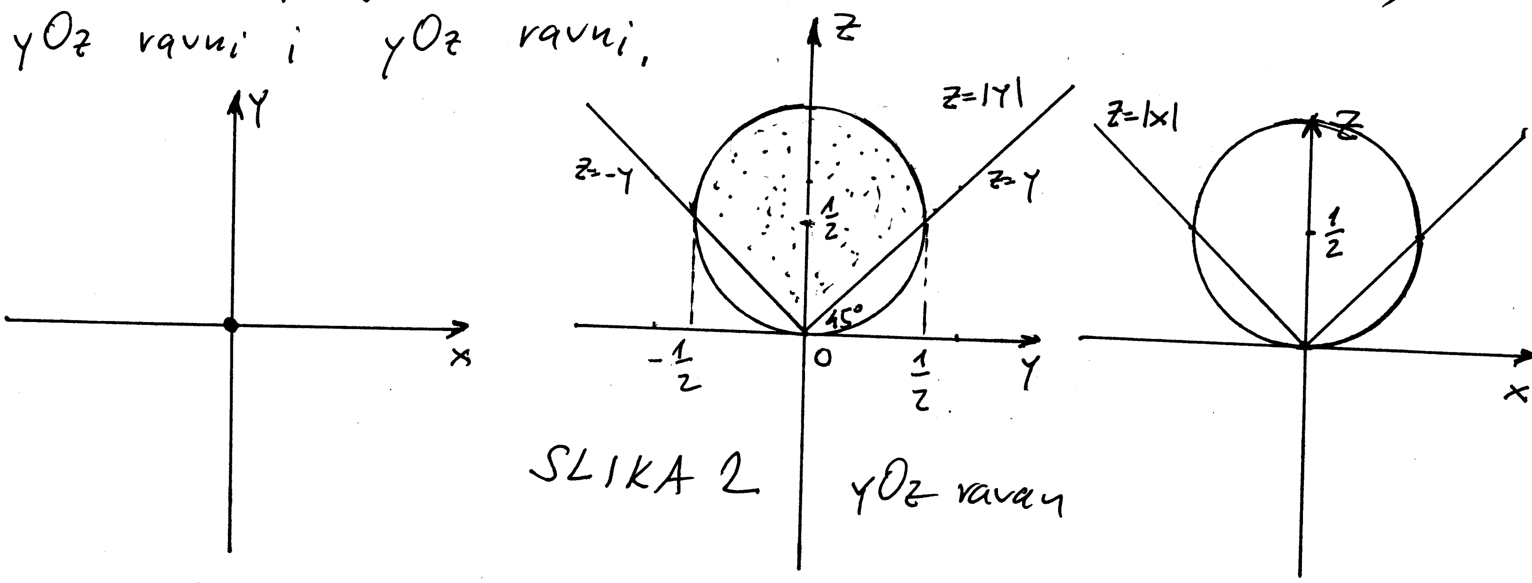
Odmah vidimo da čunj ima vrh u koordinatnom početku.

$$x^2+y^2+z^2 = z$$

$$x^2+y^2+z^2 - 2\cdot z \cdot \frac{1}{2} + \frac{1}{4} = \frac{1}{4}$$

$$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$ centar sfere je u tački $(0,0,\frac{1}{2})$ a poluprečnik $\frac{1}{2}$

Napravimo presjeke datih figura redom sa xOy ravni, yOz ravni i yOz ravni.



SLIKA 2

Sa datih slika odmah vidimo da je integral skroz teško izračunati uz pomoć pravougaonih koordinata. Uvedimo sferne koordinate.

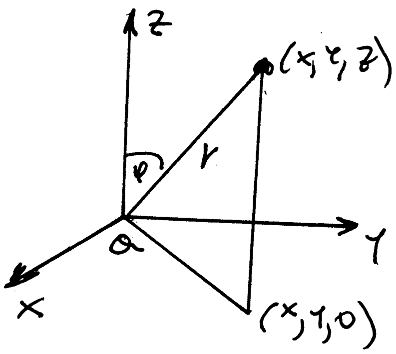
$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$

opis točke



$\Omega \xrightarrow{\text{transformiramo}} \Omega'$

Sa slike 2 čitamo granice za φ i θ. Granice za r možemo odrediti na osnovu formule

$$\Omega' : \begin{cases} 0 \le r \le \cos\varphi \\ 0 \le \varphi \le \frac{\pi}{4} \\ 0 \le \theta \le 2\pi \end{cases}$$

$x^2+y^2+z^2 \le z$

tj. $r^2 \le r\cos\varphi \quad /:r$

$r \le \cos\varphi$.

$x^2+y^2+z^2 = \dots = r^2$

$$\iiint_{\Omega} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \left| \begin{matrix} \text{uvodimo} \\ \text{sferne} \\ \text{koordinate} \end{matrix} \right| = \iiint_{\Omega'} r^3 \sin\varphi\,dr\,d\varphi\,d\theta$$

$$= \int_0^{\pi/4} \sin\varphi\,d\varphi \int_0^{\cos\varphi} r^3\,dr \int_0^{2\pi} d\theta = 2\pi \int_0^{\pi/4} \sin\varphi\,d\varphi \int_0^{\cos\varphi} r^3\,dr = 2\pi \cdot \frac{1}{4} \int_0^{\pi/4} \sin\varphi\cos^4\varphi\,d\varphi$$

$$= \left| \begin{matrix} d(\cos\varphi) = -\sin\varphi\,d\varphi \\ \sin\varphi\,d\varphi = -d(\cos\varphi) \end{matrix} \right| = -\frac{\pi}{2} \int_0^{\pi/4} \cos^4\varphi\,d(\cos\varphi) = -\frac{\pi}{2} \cdot \frac{1}{5} \cos^5\varphi \Big|_0^{\frac{\pi}{4}} =$$

$$= -\frac{\pi}{10}\left(\left(\frac{\sqrt{2}}{2}\right)^5 - 1\right) = \frac{\pi}{10}\left(1 - \frac{\sqrt{2}}{8}\right)$$ traženo rješenje

(#) Uvođenjem sfernih koordinata izračunati integral

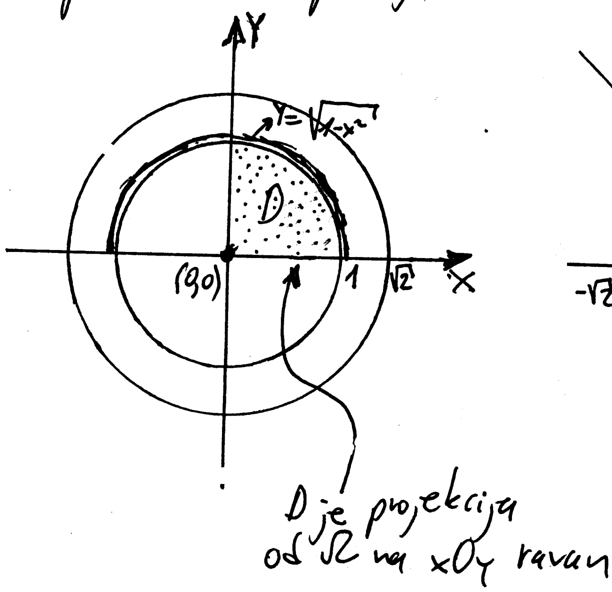
$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 dz$$

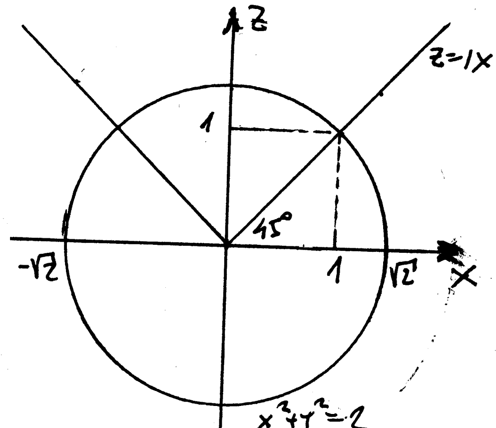
Rj. Ako sa Ω označimo oblast integracije vidimo da imamo

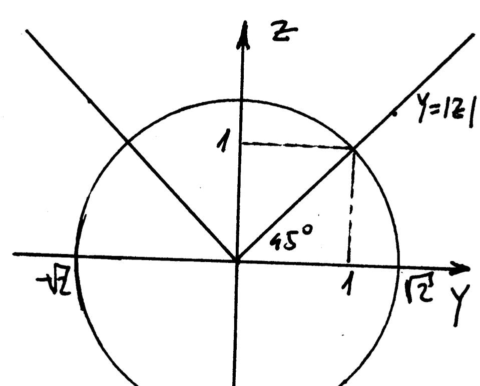
$$\Omega : \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq \sqrt{1-x^2} \\ \sqrt{x^2+y^2} \leq z \leq \sqrt{2-x^2-y^2} \end{cases}$$

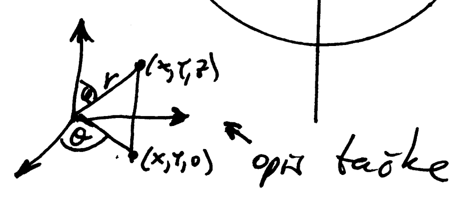
Iz $\sqrt{x^2+y^2} \leq z \leq \sqrt{2-x^2-y^2}$ čitamo da su date dvije površi $x^2+y^2=z^2$ (kupa) i $x^2+y^2+z^2=2$ (sfera)

Napravimo presjek oblasti Ω sa xOy, xOz i yOz ravni.









Ako uvedemo sferne koordinate

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$$\Omega \xrightarrow{\text{transformiše}} \Omega' : \begin{cases} 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \varphi \leq \frac{\pi}{4} \\ ??? \leq r \leq ??? \end{cases}$$

(vidimo sa projekcija)

Sa slika presjeka datih površina sa koordinatnim ravnima vidimo da će r uzimati sve tačke koje se nalaze između čunja i sfere.
Pa da bi odredili granice za r posmatrajmo formulu

$$z \leq \sqrt{2-x^2-y^2}$$

$$z^2 \leq 2-x^2-y^2$$

$$x^2+y^2+z^2 \leq 2 \quad \Rightarrow \quad r^2 \leq 2$$

$$r \leq \sqrt{2}$$

što se slaže sa nacrtanim slikama

$$\Omega' : \begin{cases} 0 \leq r \leq \sqrt{2} \\ 0 \leq \varphi \leq \frac{\pi}{4} \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$$

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2\,dz = \iiint_\Omega z^2\,dx\,dy\,dz \;\Big|\; \text{uvodimo sferne koordinate} \;\Big|\; = \iiint_{\Omega'} r^2\cos^2\varphi\, r^2\sin\varphi \; dr\,d\varphi\,d\theta$$

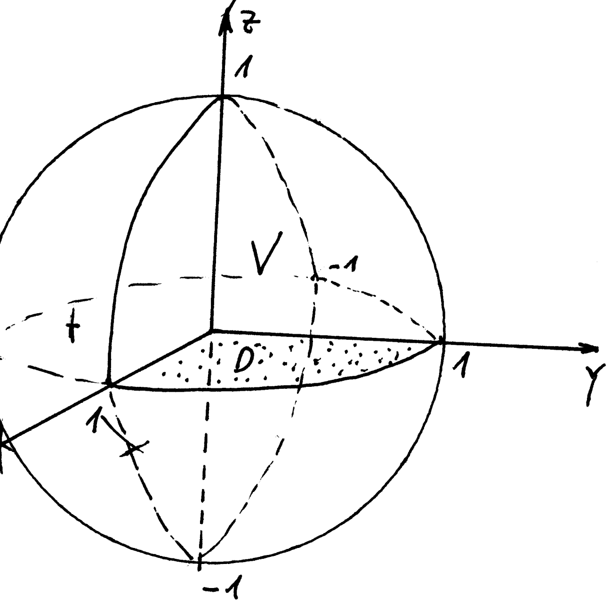
$$= \int_0^{\sqrt{2}} r^4\,dr \int_0^{\pi/4} \underbrace{\cos^2\varphi \sin\varphi\,d\varphi}_{\cos^2\varphi\,(-1)\,d(\cos\varphi)} \int_0^{\pi/2} d\theta = \theta\Big|_0^{\pi/2} \cdot \frac{-1}{3}\cos^3\varphi\Big|_0^{\frac{\pi}{4}} \cdot \frac{1}{5}r^5\Big|_0^{\sqrt{2}} =$$

$$= -\frac{1}{15}\cdot\frac{\pi}{2}\cdot 4\sqrt{2}\left(\left(\frac{\sqrt{2}}{2}\right)^3 - 1\right) = \frac{-\pi}{15}\, 2\sqrt{2}\left(\frac{2\sqrt{2}}{8} - 1\right)$$

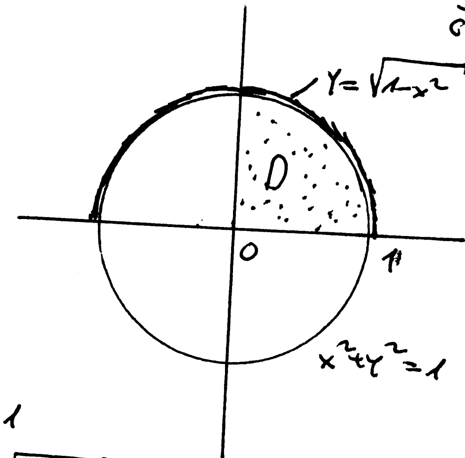
$$= \frac{\pi}{15}\left(2\sqrt{2} - 1\right)$$ traženo rješenje

(#) Izračunati integral $\iiint_V xyz\,dx\,dy\,dz$ gdje je oblast V ograničena sferom: $x^2+y^2+z^2=1$ i ravnima $x=0$, $y=0$, $z=0$ u I oktantu.

Rj. Skiciramo oblast V



$x^2+y^2+z^2=1$ predstavlja sferu sa centrom u tački $O(0,0,0)$ poluprečnika 1. Ortogonalna projekcija date sfere na xOy ravan u I oktantu je četvrtina kruga $x^2+y^2=1$.



$y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

Prema tome $V:\begin{cases} 0\le x\le 1 \\ 0\le y\le \sqrt{1-x^2} \\ 0\le z\le \sqrt{1-x^2-y^2} \end{cases}$.

Iz dobijenih granica vidimo, da bi izračunali integral, potrebno je preći na sferne koordinate

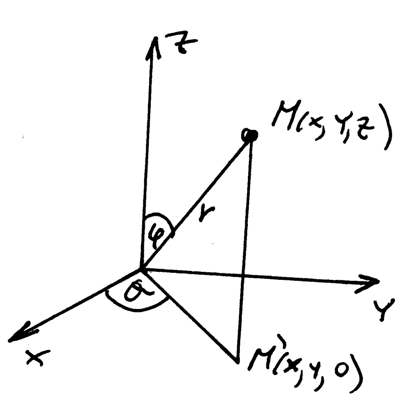
$x=r\sin\varphi\cos\theta$

$y=r\sin\varphi\sin\theta$

$z=r\cos\varphi$

$dx\,dy\,dz=r^2\sin\varphi\,dr\,d\theta\,d\varphi$

opis tačke



Uvođenjem sfernih koordinata

$V \xrightarrow{\text{transformace}} V':\begin{cases} 0\le r\le 1 \\ 0\le\varphi\le \frac{\pi}{2} \\ 0\le\theta\le\frac{\pi}{2} \end{cases}$.

Sad imamo

$$I=\iiint_V xyz\,dx\,dy\,dz = \left| \begin{array}{l} \text{uvodimo} \\ \text{sferne} \\ \text{koordinate} \end{array} \right| =$$

$$= \iiint_{V'} r\sin\varphi\cos\alpha \cdot r\sin\varphi\sin\alpha \cdot r\cos\varphi \; r^2\sin\varphi \, dr\,d\varphi\,d\alpha$$

$$= \int_0^1 r^5\,dr \int_0^{\pi/2} \sin^3\varphi\cos\varphi\,d\varphi \int_0^{\pi/2} \sin\alpha\cos\alpha\,d\alpha$$

$$\int_0^1 r^5\,dr = \frac{1}{6}r^6\Big|_0^1 = \frac{1}{6}$$

$$\int_0^{\pi/2} \sin^3\varphi\cos\varphi\,d\varphi = \left| d(\sin\varphi) = \cos\varphi\,d\varphi \right| = \int_0^{\pi/2} \sin^3\varphi\,d(\sin\varphi) =$$

$$= \frac{1}{4}\sin^4\varphi\Big|_0^{\frac{\pi}{2}} = \frac{1}{4}$$

$$\int_0^{\pi/2} \sin\alpha\cos\alpha\,d\alpha = \left| d(\sin\alpha) = \cos\alpha\,d\alpha \right| = \int_0^{\pi/2} \sin\alpha\,d(\sin\alpha) =$$

$$= \frac{1}{2}\sin^2\alpha\Big|_0^{\pi/2} = \frac{1}{2}$$

Prema tome $I = \frac{1}{6}\cdot\frac{1}{4}\cdot\frac{1}{2} = \frac{1}{48}$ traženo rješenje

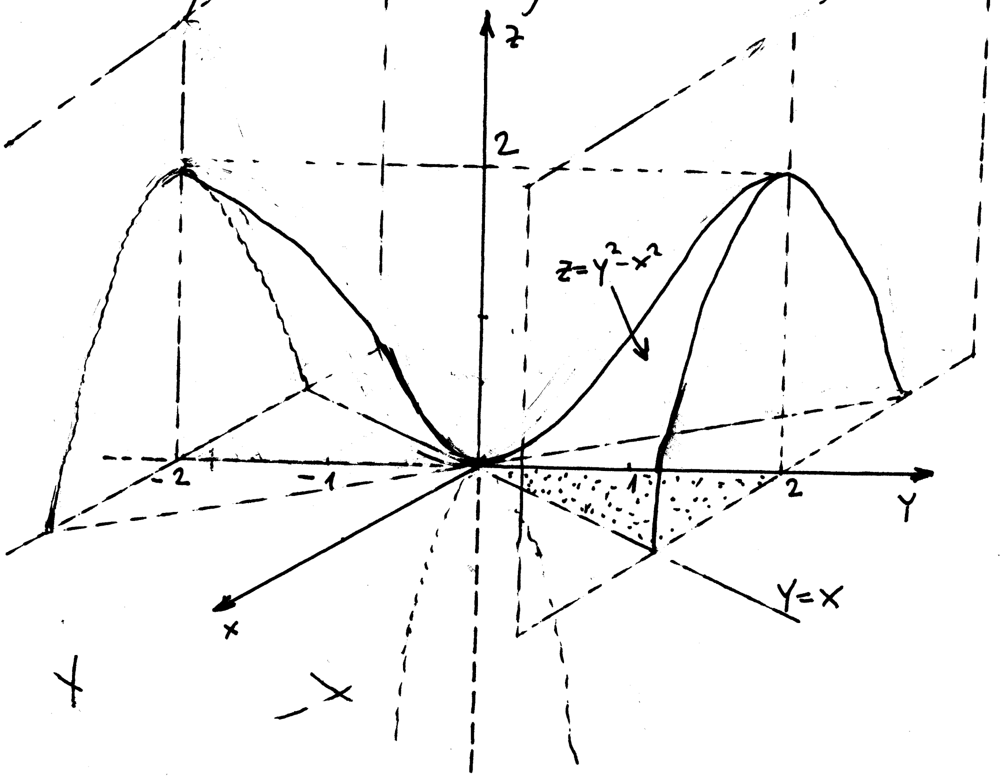
(#) Izračunati zapreminu tijela, koje je ograničeno sa površinama $z=y^2-x^2$, $z=0$, $y=\pm 2$.

Rj. Zapremina tijela se može računati pomoću dvostrukog ili pomoću trostrukog integrala. Za ta dva slučaja koristimo sljedeće dvije formule

$$V=\iint\limits_D f(x,y)\,dx\,dy\,, \qquad V=\iiint\limits_\Omega dx\,dy\,dz\,.$$

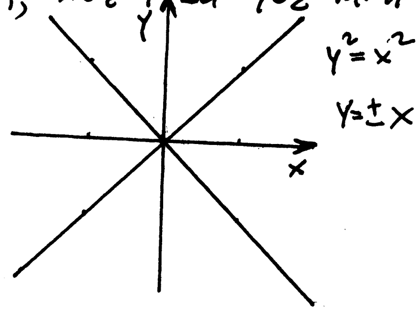
Koju od ove dvije formule je pogodnije koristiti zavisi od jednačina površina koje ograničavaju tijelo.

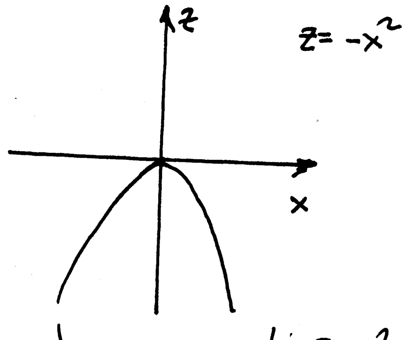
Skiciramo naše tijelo

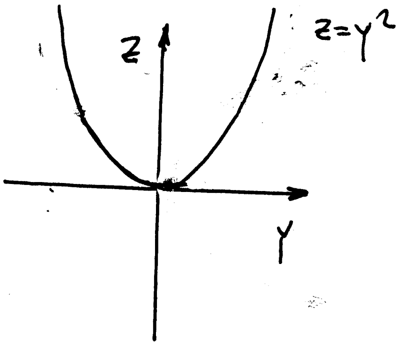


Šta predstavlja jednačina $z=y^2-x^2$?

Napravimo presjeke $z=y^2-x^2$ sa xOy, xOz i sa yOz ravni







$z(-x,-y)=(-y)^2-(-x)^2=y^2-x^2=z(x,y)$

$\Rightarrow$ tijelo je simetrično u odnosu na koordinatni početak

$z(x,-y)=(-y)^2-x^2=y^2-x^2$

$\Rightarrow$ tijelo je simetrično u odnosu na xOz osu

$z(-x,y)=y^2-(-x)^2=y^2-x^2 \Rightarrow$ tijelo je simetrično u odnosu na YoZ -osu

(#) Izračunati zapreminu tijela, ograničeno površinama

$y=x^2$, $y=1$, $x+y+z=4$, $z=0$.

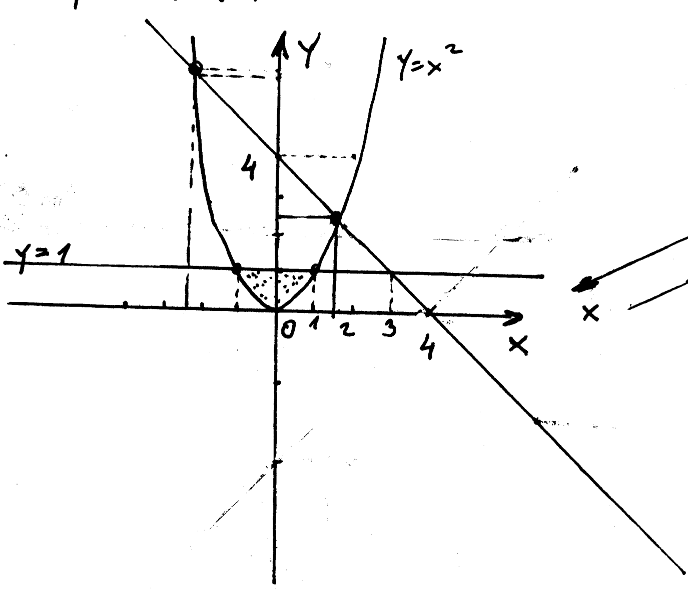
Rj. Skicirajmo naše tijelo.

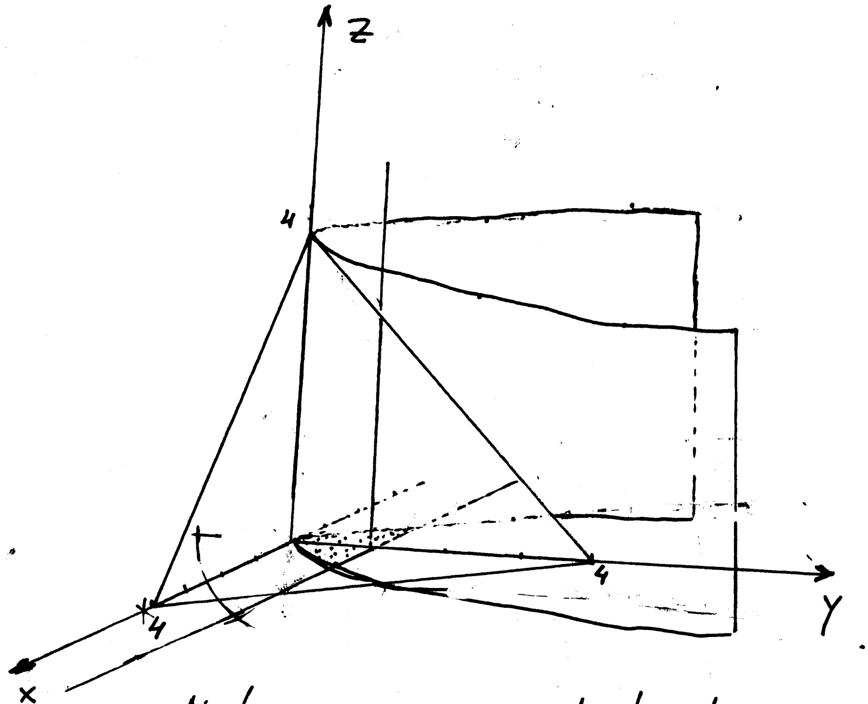
$x+y+z=4$ je ravan ($\frac{x}{4}+\frac{y}{4}+\frac{z}{4}=1$) koja na x, y, i z osi ima odsječke 4.

$y=1$, $z=0$ su ravni

$y=x^2$ je cilindar

Napravimo ortogonalne projekcije površina na xOy ravan





Nađimo presječnu tačku krive $y=x^2$ i prave $x+y=4$.

$$\begin{aligned} y&=x^2 \\ x+y&=4 \\ \hline y&=x^2 \\ y&=4-x \end{aligned} \qquad \begin{aligned} x^2&=4-x \\ x^2+x-4&=0 \\ D&=1+16=17 \\ x_{1,2}&=\frac{-1\pm 4{,}12}{2} \end{aligned} \qquad x_1\approx -2{,}56 \Rightarrow y_1=2{,}43 \qquad x_2\approx 1{,}56 \Rightarrow y_2=6{,}56$$

$V=\iint_D f(x,y)\,dx\,dy$ ← zapremina tijela koje je odozgo ograničeno sa ... i tijelo ima ortogonalnu projekciju D

U našem slučaju $f(x,y)=4-x-y$ (ovo vidimo sa skice)

$V=\iint_D (4-x-y)\,dx\,dy$ gdje je $D:\begin{cases}-1\le x\le 1\\ x^2\le y\le 1\end{cases}$ ili $D:\begin{cases}0\le y\le 1\\ -\sqrt{y}\le x\le\sqrt{y}\end{cases}$

$$V=\int_{-1}^{1}dx\int_{x^2}^{1}(4-x-y)\,dy=\int_{-1}^{1}\Big(\underbrace{4y\Big|_{x^2}^{1}}_{4(1-x^2)}-\underbrace{xy\Big|_{x^2}^{1}}_{-x(1-x^2)}-\underbrace{\tfrac{1}{2}y^2\Big|_{x^2}^{1}}_{-\frac{1}{2}(1-x^4)}\Big)dx=$$

$$=\int_{-1}^{1}\left(4-4x^2-x+x^3-\frac{1}{2}+\frac{1}{2}x^4\right)dx=\int_{-1}^{1}\left(x^3-4x^2+\frac{1}{2}x^4-x+\frac{7}{2}\right)=\ldots=-\frac{8}{3}+\frac{1}{5}+7=\frac{68}{15}$$

traženo rješenje

Izračunati zapreminu tijela ograničenog dijelom površi $(x^2+y^2+z^2)^3 = \frac{a^6 z^2}{x^2+y^2}$, $a>0$ u I oktantu.

Rj.

Zapremina tijela ograničenog sa oblasti Ω se računa po formuli $V = \iiint\limits_{\Omega} dx\,dy\,dz$.

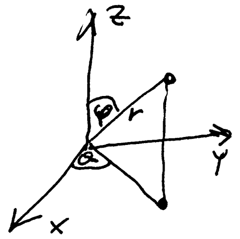
Datu površ $(x^2+y^2+z^2)^3 = \frac{a^6 z^2}{x^2+y^2}$ ne možemo skicirati.

Uvedimo sferne koordinate

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$



$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$\Omega \xrightarrow{\text{transformacije}} \Omega'$

pa pokušajmo naći granice na osnovu date formule.

$x^2+y^2+z^2 = \underline{r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi}\sin^2\theta + r^2\cos^2\varphi = r^2\sin^2\varphi + r^2\cos^2\varphi = r^2$

$(x^2+y^2+z^2)^3 = (r^2)^3 = r^6$

$z^2 = r^2\cos^2\varphi$

$x^2+y^2 = r^2\sin^2\varphi$

$(x^2+y^2+z^2)^3 = \frac{a^6 z^2}{x^2+y^2}$

sad postaje $r^6 = \frac{a^6 r^2\cos^2\varphi}{r^2\sin^2\varphi}$

tj. $r^6 = a^6\,\mathrm{ctg}^2\varphi$

$r = \sqrt[6]{a^6\,\mathrm{ctg}^2\varphi}$

$r = a\sqrt[3]{\mathrm{ctg}\varphi}$.

Na osnovu ove formule i znanja da je tijelo u I oktantu možemo zaključiti da je

$$\Omega' : \begin{cases} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le a\sqrt[3]{\mathrm{ctg}\varphi} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

$$V = \iiint\limits_{\Omega'} r^2\sin\varphi\,dr\,d\varphi\,d\theta = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\varphi\,d\varphi \int_0^{a\sqrt[3]{\mathrm{ctg}\varphi}} r^2\,dr = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\varphi \left.\frac{r^3}{3}\right|_0^{a\sqrt[3]{\mathrm{ctg}\varphi}} d\varphi$$

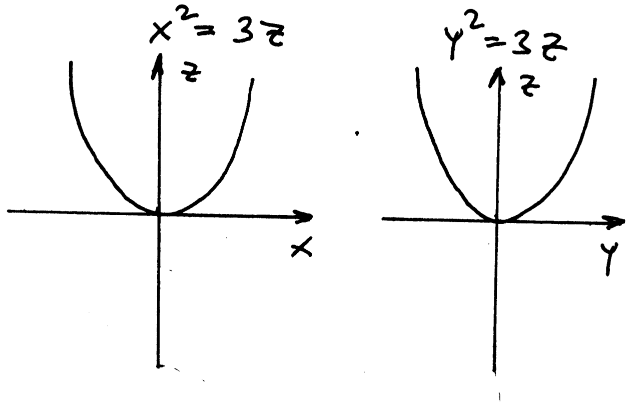
$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \frac{a^3}{3}\sin\varphi\cdot\frac{\cos\varphi}{\sin\varphi}\,d\varphi = \frac{a^3}{3}\int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos\varphi\,d\varphi = \frac{a^3}{3}\cdot\theta\Big|_0^{\pi/2}\cdot\sin\varphi\Big|_0^{\pi/2} = \frac{a^3\pi}{6}$$ tražena zapremina.

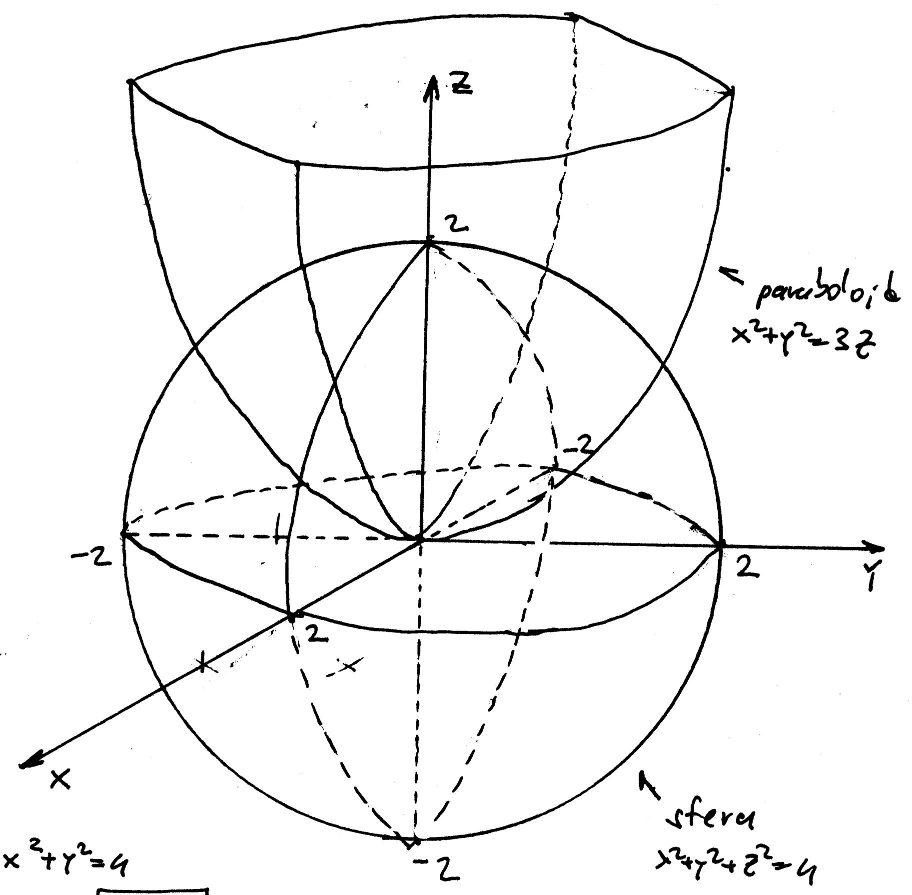
Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

Rj. $x^2+y^2+z^2=4$ je sfera sa centrom u $(0,0,0)$ poluprečnika 2

$x^2+y^2=3z$ je paraboloid

Skicirajmo ova dva tijela





$x^2+y^2=4$

$x=\pm\sqrt{4-y^2}$

$y=\pm\sqrt{4-x^2}$

$$V=\iiint_{\Omega} dx\,dy\,dz$$

Primjetimo da je tijelo dobijeno presjekom simetrično na ravni xOz i na yOz.

Prema tome

$$V=4\iiint_{\Omega_1} dx\,dy\,dz \quad \text{gdje je}$$

Ω_1 oblast u presjeku dva tijela u prvom oktantu

$$\Omega_1: \begin{cases} 0\le x\le 2 \\ 0\le y\le \sqrt{4-x^2} \\ 0\le z\le \frac{1}{3}(x^2+y^2) \end{cases}$$

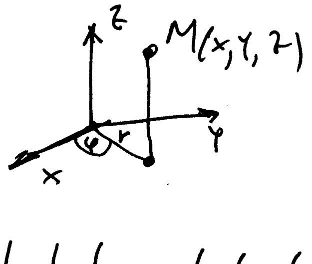
$$V=4\int_0^2 dx\int_0^{\sqrt{4-x^2}} dy\int_0^{\frac{1}{3}(x^2+y^2)} dz = 4\int_0^2 dx\int_0^{\sqrt{4-x^2}} \frac{1}{3}(x^2-y^2)\,dy$$

$$=\frac{4}{3}\int_0^2 \left(x^2 y\Big|_0^{\sqrt{4-x^2}} + \frac{1}{3}y^3\Big|_0^{\sqrt{4-x^2}} \right) dx = \ldots = \frac{8\pi}{3}$$

komplikovano

II način:

Uvedimo cilindrične koordinate
$x=r\cos\varphi$
$y=r\sin\varphi$
$z=z$



$dx\,dy\,dz=r\,dr\,d\varphi\,dz$

Oblast Ω_1 $\overset{\text{transformise}}{\rightsquigarrow}$ Ω_1' : $\begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le \frac{1}{3} r^2 \end{cases}$

$$V = 4 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2} dr \int_0^{\frac{1}{3}r^2} r\, dz = 4 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2} r \cdot \frac{1}{3} r^2 \, dr = \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big|_0^2 d\varphi = \frac{1}{3} \cdot 16 \cdot \frac{\pi}{2} = \frac{8\pi}{3}$$

$V = \dfrac{8\pi}{3}$ tražena zapremina

(#) Izračunati zapreminu tijela ograničenog valjkom $x^2+y^2=6x$ i ravnima $x-z=0$, $5x-z=0$.

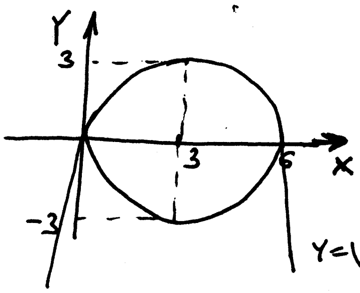
Rj. $V=\iiint\limits_{\Omega} dx\,dy\,dz$

$x^2+y^2=6x$

$x^2-2\cdot x\cdot 3+3^2-3^2+y^2=0$

$(x-3)^2+y^2=3^2$

projekcija valjka na xOy ravan izgleda



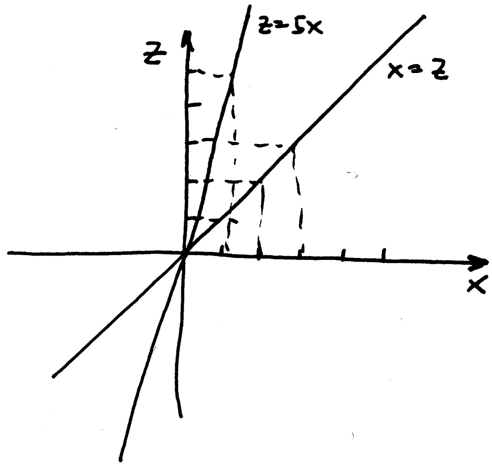
$y^2=6x-x^2$

$y=\pm\sqrt{6x-x^2}$

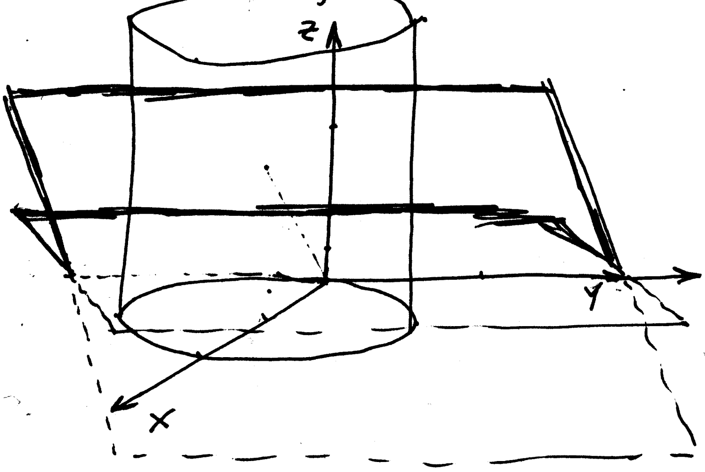
$x-z=0$ $\quad$ $5x-z=0$

$x=z$ $\quad$ $z=5x$

projekcije ravni $x-z=0$ i $5x-z=0$ na xOz ravan izgleda



Skica ovih figura u prostoru bi otprilike izgledala ovako

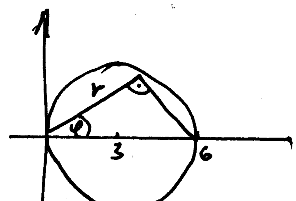


valjak presječen sa dvije ravni na klasičan način

$$\Omega:\begin{cases}0<x<6\\ 0<y<\sqrt{6x-x^2}=\sqrt{9-(x-3)^2}\\ x\le z\le 5x\end{cases}$$

uvedimo cilindrične koordinate

$x=r\cos\varphi$
$y=r\sin\varphi$
$z=z$
$dx\,dy=r\,dr\,d\varphi$



$\cos\varphi=\frac{r}{6}$

Primjetimo da je oblast Ω simetrična u odnosu na xOz ravan

$$V=2\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz \qquad \Omega':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 6\cos\varphi\\ r\cos\varphi<z<5r\cos\varphi\end{cases}$$

$$V=2\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz=2\int_0^{\pi/2}d\varphi\int_0^{6\cos\varphi}r\,dr\int_{r\cos\varphi}^{5r\cos\varphi}dz=8\int_0^{\pi/2}\cos\varphi\,d\varphi\int_0^{6\cos\varphi}r^2\,dr=8\int_0^{\pi/2}\frac{1}{3}r^3\Big|_0^{6\cos\varphi}\cos\varphi\,d\varphi$$

($\int_{r\cos\varphi}^{5r\cos\varphi}dz = 4r\cos\varphi$)

$$=8\cdot 6^3/3\int_0^{\pi/2}\cos^4\varphi\,d\varphi=576\int_0^{\pi/2}\left(\frac{1}{2}(1+\cos2\varphi)\right)^2d\varphi=144\int_0^{\pi/2}(1+2\cos2\varphi+\cos^2 2\varphi)\,d\varphi=\ldots=108\pi$$

za vježbu

tražena zapremina ↑

(#) Izračunati zapreminu tijela ograničenog ravninom XOY, valjkom $x^2+y^2=2ax$ i čunjem $x^2+y^2=z^2$.

Rj. Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi $V=\iiint\limits_{\Omega} dx\,dy\,dz$. Pokušajmo skicirati tijelo čiju zapreminu tražimo.

valjak $x^2+y^2=2ax$

$$x^2-2ax+y^2=0$$

$$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$$

$$(x-a)^2+y^2=a^2$$

valjak u presjeku sa XOY ravni je krug sa centrom u tački $(a,0)$ poluprečnika a

čunj $x^2+y^2=z^2$ u presjeku sa XOY ravni je tačka, a u presjeku sa YOZ ili sa XOZ su po dvije prave

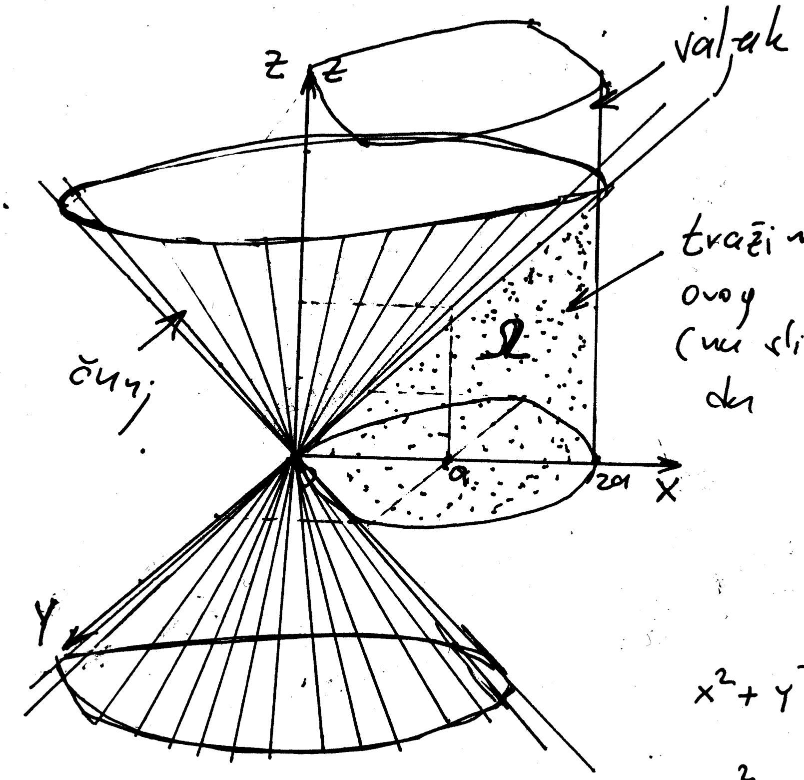
Oblast Ω je najlakše projicirati na XOY ravan.

Uvodimo cilindrične koordinate

$$x=a+r\cos\varphi$$

$$y=r\sin\varphi$$

$$z=z$$



tražimo zapreminu ovog tijela (na slici smo pretpostavili da je $a>0$)

$$\Omega': \begin{cases} dx\,dy\,dz=r\,dr\,d\varphi\,dz \\ 0\le r\le a \\ 0\le\varphi\le 2\pi \end{cases}$$

$$x^2+y^2=z^2$$

$$z=\pm\sqrt{x^2+y^2} \quad \text{čunj}$$

$$x^2+y^2=(a+r\cos\varphi)^2+(r\sin\varphi)^2=$$

$$=a^2+2ar\cos\varphi+r^2\cos^2\varphi+r^2\sin^2\varphi=$$

$$=a^2+2ar\cos\varphi+r^2$$

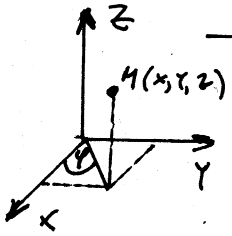
$$V=\iiint\limits_{\Omega} dx\,dy\,dz=\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz=\int_0^a dr\int_0^{2\pi} d\varphi\int_0^{\sqrt{a^2+2ar\cos\varphi+r^2}} r\,dz = \ldots$$ ovo je teško izračunati

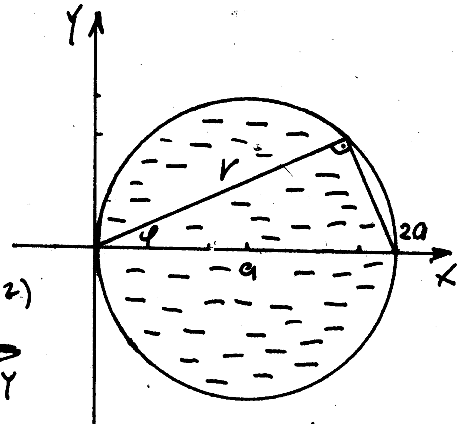
Pokušajmo uvesti drugačije smjene.

$x = r\cos\varphi$ $\qquad dx\,dy\,dz = r\,dr\,d\varphi\,dz$

$y = r\sin\varphi$

$z = z$ $\qquad x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$





$$\Omega'' : \begin{cases} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2a\cos\varphi \\ 0 \le z \le \sqrt{r^2} \end{cases}$$

$\cos\varphi = \dfrac{r}{2a}$

$r = 2a\cos\varphi$

$$V = \iiint\limits_{\Omega} dx\,dy\,dz = \iiint\limits_{\Omega''} r\,dr\,d\varphi\,dz =$$

$$= \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} dr \int_0^{r} r\,dz = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} \left(r z \Big|_0^r \right) dr = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3} r^3 \Big|_0^{2a\cos\varphi} \right) d\varphi = \int_{-\pi/2}^{\pi/2} \frac{8}{3} a^3 \cos^3\varphi\,d\varphi = \frac{8}{3} a^3 \int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi$$

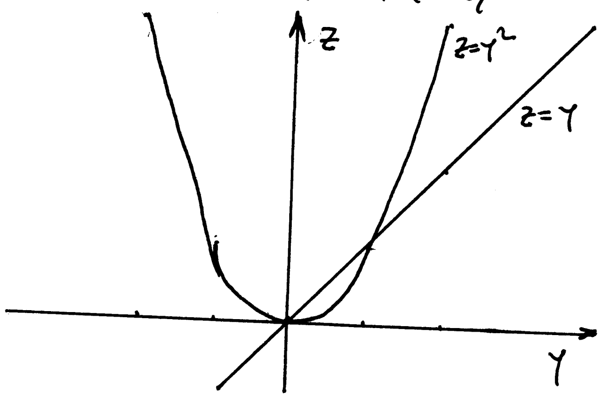
$$\int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi\cos^2\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi\,(1-\sin^2\varphi)\,d\varphi = \left| \begin{array}{l} \sin\varphi = t \\ \cos\varphi\,d\varphi = dt \\ \varphi = -\frac{\pi}{2} \Rightarrow t = -1 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{array} \right|$$

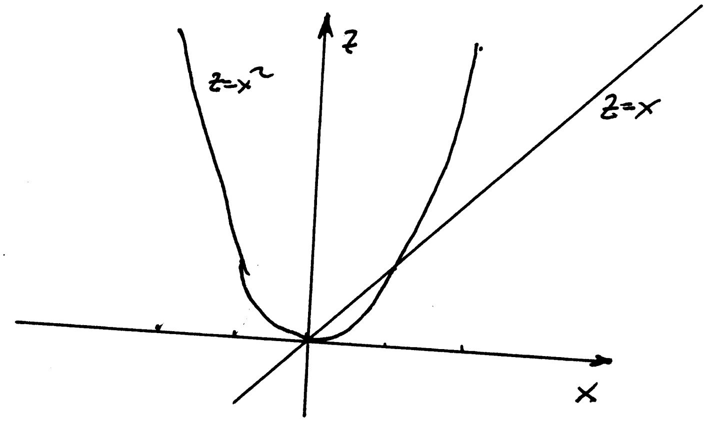
$$= \int_{-1}^{1} (1-t^2)\,dt = t\Big|_{-1}^{1} - \frac{1}{3}t^3\Big|_{-1}^{1} = 2 - \frac{1}{3}\cdot 2 = \frac{4}{3}$$

$V = \dfrac{32}{9} a^3$ tražena zapremina

Izračunati zapreminu tijela koju ravan $z=x+y$ odsijeca od paraboloida $z=x^2+y^2$.

Rj. Pogledajmo kako izgleda presjek datih površina sa yOz i xOz ravni





Na osnovu ove dvije slike pokušajte skicirati tijelo u prostoru!

$$V=\iiint\limits_{\Omega} dx\,dy\,dz = \iint\limits_{D} dx\,dy\int\limits_{x^2+y^2}^{x+y} dz = \iint\limits_{D}\left(x+y-(x^2+y^2)\right)dx\,dy \overset{(*)}{=}$$

gdje je D ortogonalna projekcija datog tijela na xOy ravan.

Projekciju presjeka tijela određujemo na sljedeći način

$$\begin{aligned} z&=x+y\\ z&=x^2+y^2 \\ \hline \end{aligned}$$

$$x+y=x^2+y^2 \Rightarrow x^2-x+y^2-y=0$$

$$x^2-2\cdot x\cdot\frac{1}{2}+\frac{1}{4}+y^2-2\cdot y\cdot\frac{1}{2}+\frac{1}{4}=\frac{1}{2}$$

$$D:\ \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$$

Ako uvedemo polarne koordinate $x=\frac{1}{2}+r\cos\varphi$, $y=\frac{1}{2}+r\sin\varphi$, $dx\,dy=r\,dr\,d\varphi$

$D \xrightarrow{\text{transformire}} D'$

$$D': \begin{cases} 0 \leq r \leq \frac{1}{\sqrt{2}} \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$\overset{(*)}{=} \iint_D (-1)(x^2 - x + y^2 - y)\, dx\, dy = (-1) \iint_D \left(\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2 - \frac{1}{2} \right) dx\, dy =$$

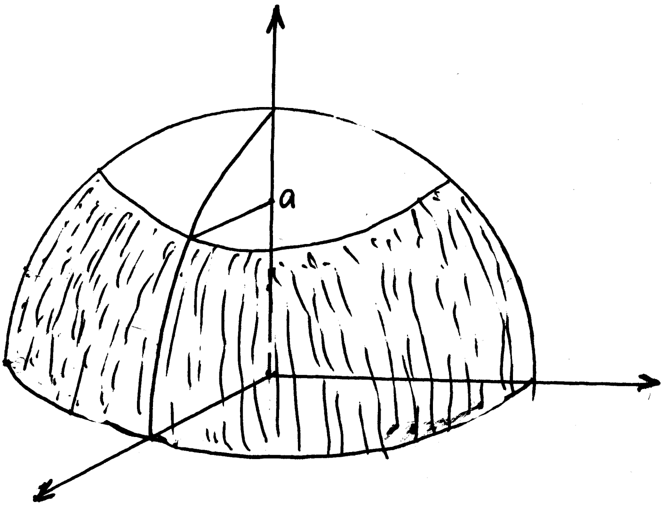
Primjetimo da je $x - \frac{1}{2} = r \cos\varphi$

$y - \frac{1}{2} = r \sin\varphi$

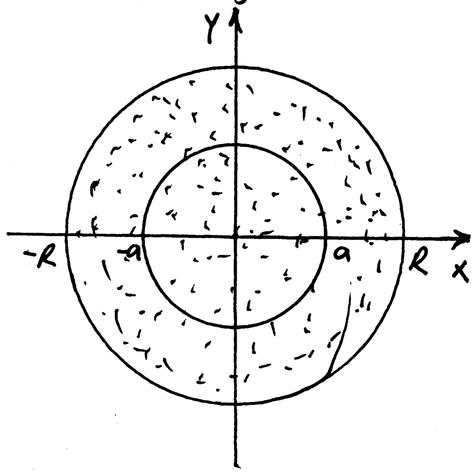
$$= (-1) \iint_{D'} \left(r^2 - \frac{1}{2}\right) r\, dr\, d\varphi = (-1) \int_0^{2\pi} d\varphi \int_0^{1/\sqrt{2}} \left(r^3 - \frac{1}{2} r\right) dr = \ldots = \frac{\pi}{8}$$

traženo rješenje

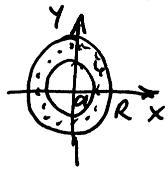
(#) Izračunati zapreminu dijela kugle $x^2+y^2+z^2=R^2$ koji se nalazi između dvije paralelne ravni $z=0$ i $z=a$ $(0<a<R)$.



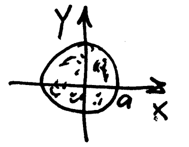
Ortogonalna projekcija figure na xOy ravan izgleda



Sa slike vidimo, da bi odredili zapreminu V pronaći ćemo dvije zapremine V_1 i V_2 i to za figure Ω_1 i Ω_2 čije će ortogonalne projekcije (na xOy ravan) biti:

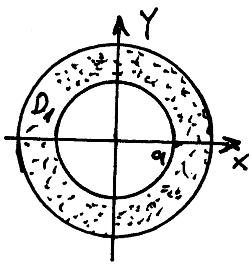


i



. Na kraju

$$V = V_1 + V_2$$



$$V_1 = \iint\limits_{D_1} f_1(x,y)\,dx\,dy = \left|\begin{array}{l} \text{u ovom slučaju} \\ z^2 = R^2 - x^2 - y^2 \\ z = \pm\sqrt{R^2-x^2-y^2} \\ \text{nama treba } +\sqrt{} \end{array}\right.$$

polarne koordinate
$x = r\cos\varphi$
$y = r\sin\varphi$
$dx\,dy = r\,dr\,d\varphi$

$D_1 \xrightarrow{\text{transf.}} D_1' : \begin{cases} a \le r \le R \\ 0 \le \varphi < 2\pi \end{cases}$

$$= \int_0^{2\pi} d\varphi \int_a^R \sqrt{R^2-r^2}\,dr \overset{(*)}{=}$$

$$\int \sqrt{R^2-x^2}\,dx = \left| \begin{array}{ll} u = \sqrt{R^2-x^2} & dv = dx \\ du = \dfrac{-x}{\sqrt{R^2-x^2}} & v = x \end{array} \right| = x\sqrt{R^2-x^2} + \int \frac{x^2 \overbrace{-R^2+R^2}^{-(R^2-x^2)+R^2}}{\sqrt{R^2-x^2}}\,dx =$$

$$= x\sqrt{R^2-x^2} - \int \frac{R^2-x^2}{\sqrt{R^2-x^2}} + R^2 \int \frac{dx}{\sqrt{R^2-x^2}} = x\sqrt{R^2-x^2} - \int\sqrt{R^2-x^2} + R^2 \arcsin\frac{R}{a} + C_1$$

$$\Rightarrow \int \sqrt{R^2-x^2}\,dx = \frac{1}{2}x\sqrt{R^2-x^2} + \frac{1}{2}R^2 \arcsin\frac{x}{R} + C$$

$$\overset{(*)}{=} \int_0^{2\pi} \left(\frac{1}{2}x\sqrt{R^2-x^2}\Big|_a^R + \frac{1}{2}R^2\arcsin\frac{x}{R}\Big|_a^R \right) d\varphi =$$

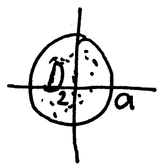
$$= \Bigg(-\frac{1}{2}a\sqrt{R^2-a^2} + \frac{1}{2}R^2\Big(\underbrace{\arcsin 1}_{\frac{\pi}{2}} - \arcsin\frac{a}{R}\Big)\Bigg)2\pi$$

$$= -a\sqrt{R^2-a^2}\,\pi + \frac{\pi^2 R^2}{2} - R^2\arcsin\frac{a}{R}\,\pi$$

$$V_2 = \iint_{D_2} f_2(x,y)\,dx\,dy =$$

u ovom slučaju

$f_2(x,y) = z = a$



polarne koordinate

$x = r\cos\varphi$

$y = r\sin\varphi$

$dx\,dy = r\,dr\,d\varphi$

$$D_2 \xrightarrow{\text{transf.}} D_2' : \begin{cases} 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$= \int_0^{2\pi} d\varphi \int_0^a r\,dr\,d\varphi = \frac{1}{2}r^2\Big|_0^a \cdot \varphi\Big|_0^{2\pi} = a^2\pi$$

$$V = V_1 + V_2 = a^2\pi - a\sqrt{R^2-a^2}\,\pi + \frac{\pi^2R^2}{2} - \pi R^2\arcsin\frac{a}{R}$$

tražena zapremina

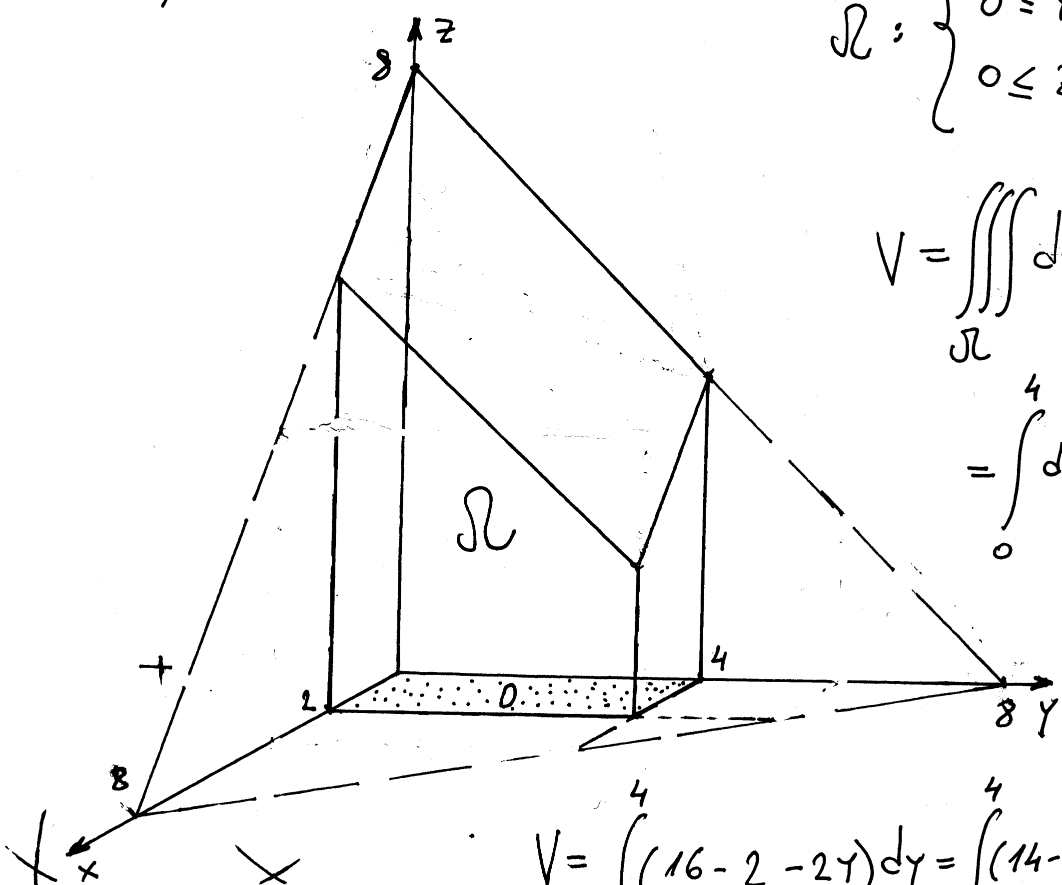
(#) Naći težište homogenog tijela ograničenog sa ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasječen paralelopiped).

Rj. Težište $T(x_T, y_T, z_T)$ homogenog tijela ograničenog sa oblašću Ω tražimo po formulama

$$x_T = \frac{1}{V}\iiint\limits_{\Omega} x\,dx\,dy\,dz, \quad y_T = \frac{1}{V}\iiint\limits_{\Omega} y\,dx\,dy\,dz, \quad z_T = \frac{1}{V}\iiint\limits_{\Omega} z\,dx\,dy\,dz$$

gdje je V zapremina tijela Ω.

Skicirajmo dato tijelo



$$\Omega : \begin{cases} 0 \le x \le 2 \\ 0 \le y \le 4 \\ 0 \le z \le 8-x-y \end{cases}$$

$$V = \iiint\limits_{\Omega} dx\,dy\,dz = \iint\limits_{D} (8-x-y)\,dx\,dy$$

$$= \int_0^4 dy \int_0^2 (8-x-y)\,dx =$$

$$= \int_0^4 \left(8x\Big|_0^2 - \frac{1}{2}x^2\Big|_0^2 - yx\Big|_0^2\right) d$$

$$V = \int_0^4 (16-2-2y)\,dy = \int_0^4 (14-2y)\,dy = 14y\Big|_0^4 - 2\cdot\frac{1}{2}y^2\Big|_0^4$$

$$V = 14\cdot 4 - 16 = 4(14-4) = 40$$

$$V = 40$$

$$\iiint\limits_{\Omega} x\,dx\,dy\,dz = \int_0^4 dy \int_0^2 x\,dx \int_0^{8-x-y} dz = \int_0^4 dy \int_0^2 (8x - x^2 - yx)\,dx = \int_0^4 \left(4x^2\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2 - y\frac{1}{2}x^2\Big|_0^2\right) dy$$

$$= \int_0^4 \left(16 - \frac{8}{3} - 2y\right) dy = \int_0^4 \left(\frac{40}{3} - 2y\right) dy = \frac{40}{3}y\Big|_0^4 - 2\cdot\frac{1}{2}y^2\Big|_0^4 = \frac{160}{3} - 16 = \frac{112}{3}$$

$$\iiint_\Omega y\,dx\,dy\,dz = \int_0^2 dx \int_0^4 y\,dy \int_0^{8-x-y} dz = \int_0^2 dx \int_0^4 y(8-x-y)\,dy = \int_0^2 dx \int_0^4 (8y - xy - y^2)\,dy =$$

$$= \int_0^2 \left(8\frac{1}{2}y^2\Big|_0^4 - x\frac{1}{2}y^2\Big|_0^4 - \frac{1}{3}y^3\Big|_0^4\right) dx = \int_0^2 \left(64 - 8x - \frac{64}{3}\right) dx = \int_0^2 \left(\frac{128}{3} - 8x\right) dx =$$

$$= \frac{128}{3}x\Big|_0^2 - 8\cdot\frac{1}{2}x^2\Big|_0^2 = \frac{256}{3} - 16 = \frac{208}{3}$$

$$\iiint_\Omega z\,dx\,dy\,dz = \ldots = \frac{320}{3}$$ (završiti za vježbu)

Prema tome, $x_T = \frac{1}{V}\iiint_\Omega x\,dx\,dy\,dz = \frac{1}{40}\cdot\frac{112}{3} = \frac{14}{15}$

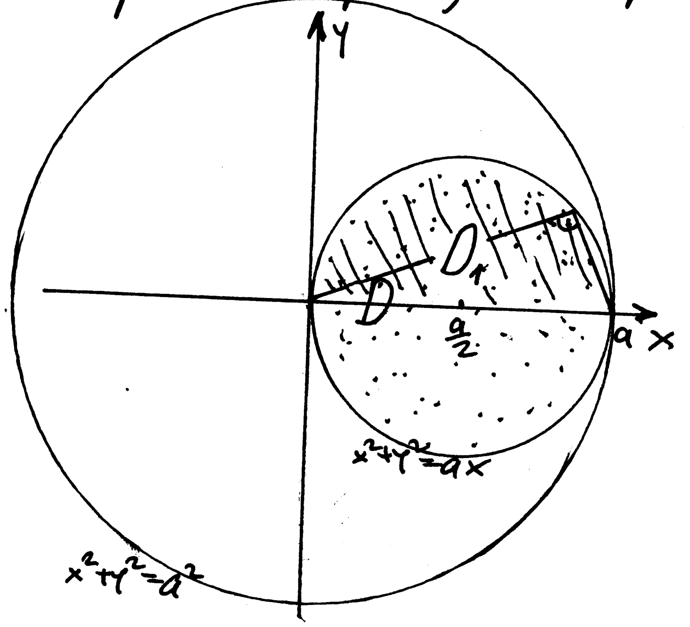
$$y_T = \frac{1}{V}\iiint_\Omega y\,dx\,dy\,dz = \frac{1}{40}\cdot\frac{208}{3} = \frac{25}{15}$$

$$z_T = \frac{1}{V}\iiint_\Omega z\,dx\,dy\,dz = \frac{1}{40}\cdot\frac{320}{3} = \frac{8}{3}$$

Težište homogenog tijela je $T\left(\frac{14}{15}, \frac{25}{15}, \frac{8}{3}\right)$.

\# Izračunati zapreminu tijela ograničenog loptom $x^2+y^2+z^2=a^2$, cilindrom $x^2+y^2=ax$ i ravni Oxy koji se nalazi u gornjem poluprostoru.

Rj. Napravimo presjek lopte i cilindra sa ravni xOy.



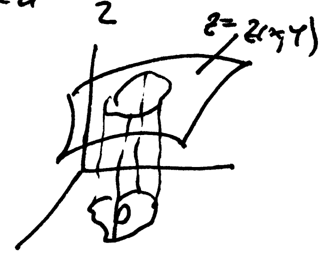
$$x^2+y^2=ax$$

$$x^2-2\cdot x\cdot\frac{a}{2}+\left(\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

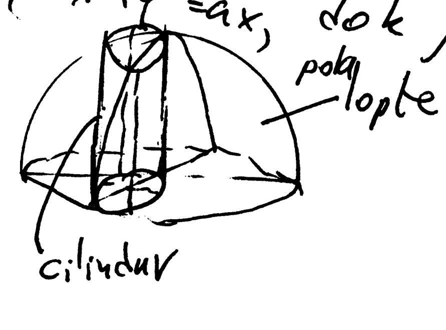
$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

krug sa centrom $C(\frac{a}{2},0)$ poluprečnika $\frac{a}{2}$

$$V=\iint_D z(x,y)\,dx\,dy$$



U našem slučaju D je unutrašnjost kruga $x^2+y^2=ax$, dok je $z(x,y)$ dio lopte iznad xOy ravni.



$$z^2=a^2-x^2-y^2$$

$$z=\pm\sqrt{a^2-x^2-y^2} \qquad \text{nama treba } +$$

$$V=\iint_D\sqrt{a^2-x^2-y^2}\,dx\,dy=2\iint_{D_1}\sqrt{a^2-x^2-y^2}\,dx\,dy=\left|\begin{array}{l}\text{uvedimo cilindrične koordinate}\\ x=\rho\cos\varphi\\ y=\rho\sin\varphi\\ dx\,dy=\rho\,d\rho\,d\varphi\end{array}\right| \quad D_1\xrightarrow{\text{transf.}}D_1':\left\{\right.$$

$$\cos\varphi=\frac{\rho}{a}$$

$$\left. D_1':\begin{cases}0\le\rho\le a\cos\varphi\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}\right| =2\iint_{D_1'}\sqrt{a^2-\rho^2}\,\rho\,d\rho\,d\varphi=\left|\begin{array}{l}d(a^2-\rho^2)=-2\rho\,d\rho\\ \rho\,d\rho=-\frac{1}{2}d(a^2-\rho^2)\end{array}\right|=$$

$$=2\cdot\frac{-1}{2}\int_0^{\pi/2}d\varphi\int_0^{a\cos\varphi}\sqrt{a^2-\rho^2}\,d(a^2-\rho^2)=-\int_0^{\pi/2}\frac{2}{3}(a^2-\rho^2)^{\frac{3}{2}}\Big|_0^{a\cos\varphi}d\varphi=-\frac{2}{3}\int_0^{\pi/2}(\sin^3\varphi-1)\,d\varphi\overset{\text{za vjež.}}{=}\ldots=\frac{a^3}{9}(3\pi-4)$$

(#) Izračunati krivolinijski integral $I=\int\limits_L (4\sqrt[3]{x}-3\sqrt{y})\,dl$

između tački $E(-1;0)$ i $F(0;1)$

a) po pravoj EF;

b) po liniji astroide $x=\cos^3 t$, $y=\sin^3 t$.

Rj.

$$I=\int\limits_L (4\sqrt[3]{x}-3\sqrt{y})\,dl$$

Ovo je krivolinijski integral prve vrste. Prisjetimo se

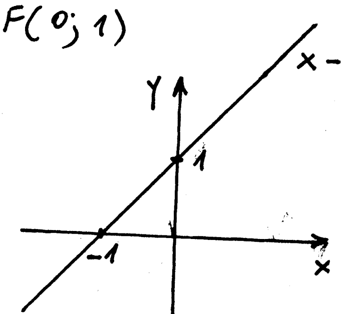
Ako je L kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$ tada

$$\int\limits_L f(x,y)\,dl=\int\limits_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

Ako je L opisana parametarskim jednačinama $\begin{cases} x=\mu(t) \\ y=\theta(t)\end{cases}$ gdje $t_1\le t\le t_2$

$$\int\limits_L f(x,y)\,dl=\int\limits_{t_1}^{t_2} f(\mu(t),\theta(t))\sqrt{(\mu'(t))^2+(\theta'(t))^2}\,dt$$

a) $E(-1;0)$
$F(0;1)$



$-y=-x-1$, $x\in[-1,0]$ tj. $y=x+1$

$y'=1 \Rightarrow dl=\sqrt{1+1^2}\,dx=\sqrt{2}\,dx$

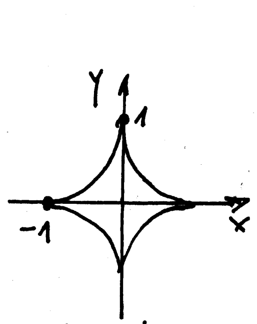
$$I=\int\limits_L (4\sqrt[3]{x}-3\sqrt{y})\,dl=\int\limits_{-1}^{0}(4x^{\frac{1}{3}}-3(x+1)^{\frac{1}{2}}\sqrt{2}\,dx$$

$$=4\sqrt{2}\int\limits_{-1}^{0}x^{\frac{1}{3}}dx-3\sqrt{2}\int\limits_{-1}^{0}(x+1)\,dx=$$

$$=4\sqrt{2}\cdot\frac{3}{4}x^{\frac{4}{3}}\Big|_{-1}^{0}-3\sqrt{2}\int\limits_{-1}^{0}(x+1)^{\frac{1}{2}}d(x+1)=3\sqrt{2}\,(0-1)-3\sqrt{2}\cdot\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_{-1}^{0}=-5\sqrt{2}$$

↑ traženo rješenje

b)



astroida

$x=\cos^3 t$, $x'=-3\cos^2 t\sin t$

$y=\sin^3 t$, $y'=3\sin^2 t\cos t$

$$dl=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt$$

$$\sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3\sqrt{\cos^2 t \sin^2(t)\underbrace{(\cos^2 t + \sin^2 t)}_{=1}} =$$

$$= 3\,|\sin t \cos t|$$

U našem slučaju t uzima vrijednost od $\frac{\pi}{2}$ do π, pa je

$$dl = -3\sin t \cos t\, dt$$

$$I = \int_L (4\sqrt[3]{x} - 3\sqrt{y})\, dl = \int_{\frac{\pi}{2}}^{\pi} \left(4\sqrt[3]{\cos^3 t} - 3\sqrt{\sin^3 t}\right)(-3\sin t \cos t)\, dt$$

$$= -12\int_{\frac{\pi}{2}}^{\pi} \cos^2 t \sin t\, dt + 9\int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{5}{2}} t \cos t\, dt =$$

$$= +12\int_{\frac{\pi}{2}}^{\pi} \cos^2 t\, d\cos t + 9\int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{5}{2}} t\, d\sin t = 12\cdot \frac{\cos^3 t}{3}\Big|_{\frac{\pi}{2}}^{\pi} + 9\cdot \frac{\sin^{\frac{7}{2}} t}{\frac{7}{2}}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 4\left((-1)^3 - 0\right) + \frac{18}{7}\left(0 - 1^{\frac{7}{2}}\right) = -4 - \frac{18}{7} = -\frac{46}{7}$$ traženo rješenje

(#) Izračunati krivolinijski integral prve vrste

$$I = \oint_C \sqrt{x^2+y^2}\, ds$$

gdje je C krug $x^2+y^2 = ax$ $(a>0)$.

Rj. Prisjetimo se

Ako je c kriva opisana parametarski $c: \begin{cases} x=\eta(t) \\ y=\mu(t) \\ t_1 \le t \le t_2 \end{cases}$ tada

$$\int_c f(x,y)\, ds = \int_{t_1}^{t_2} f(\eta(t), \mu(t)) \underbrace{\sqrt{(\eta'(t))^2 + (\mu'(t))^2}\, dt}_{ds}$$

$x^2+y^2 = ax$

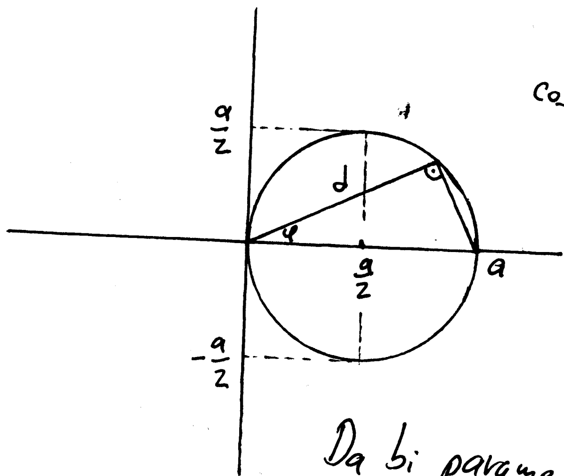
$x^2 - ax + y^2 = 0$

$x^2 - 2\cdot x \cdot \frac{a}{2} + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$

$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$

krug sa centrom $C(\frac{a}{2}, 0)$

poluprečnika $r = \frac{a}{2}$



$\cos\varphi = \frac{d}{a}$

$d = a\cos\varphi$

Da bi parametrizirali dati krug pomoći će nam polarne koordinate

$x = r\cos\varphi$

$y = r\sin\varphi$

Kako r zavisi od ugla imam

$r = a\cos\varphi$

$\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Parametrizacija datog kruga je

$\left.\begin{array}{l} x = a\cos\varphi\cos\varphi = a\cos^2\varphi \\ y = a\cos\varphi\sin\varphi = \frac{a}{2}\sin 2\varphi \end{array}\right\} \Rightarrow x^2+y^2 = a^2\cos^2\varphi$

$\left.\begin{array}{l} x'_t = 2a\cos\varphi(-\sin\varphi) = -2a\sin\varphi\cos\varphi = -a\sin 2\varphi \\ y'_t = a\cos 2\varphi \end{array}\right\} \Rightarrow \sqrt{x'^2_t + y'^2_t} = \sqrt{a^2(\sin^2 2\varphi + \cos^2 2\varphi)} = a$

$$I = \oint_C \sqrt{x^2+y^2}\, ds = \int_{-\pi/2}^{\pi/2} a\cos\varphi \cdot a\, d\varphi = a^2 \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi = a^2 \sin\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2a^2$$ traženo rješenje

(#) Izračunati krivoliniski integral $\int_L (x-y)\,ds$ po kružnoj liniji $x^2+y^2=ax$.

Rj.

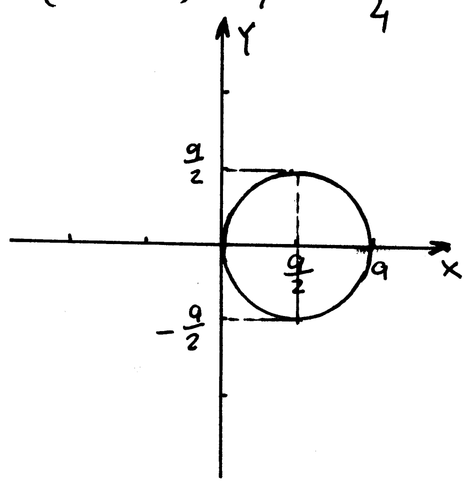
$$x^2+y^2=ax$$

$$x^2-ax+y^2=0$$

$$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$

krug sa centrom u $C(\frac{a}{2},0)$ poluprečnika $r=\frac{a}{2}$



Kako se računa krivoliniski integral $\int_L f(x,y)\,ds$?

Ako je kriva L data u obliku f-je $y=\eta(x)$ gdje je $a\le x\le b$ tada

$$\int_L f(x,y)\,ds=\int_a^b f(x,\eta(x))\sqrt{1+(y')^2}\,dx.$$

Ako je kriva L data u parametarskom obliku

$$\begin{cases} x=\mu(t)\\ y=\theta(t)\\ t_1\le t\le t_2\end{cases}$$

tada $\int_L f(x,y)\,ds=\int_{t_1}^{t_2} f(\mu(t),\theta(t))\sqrt{\mu'(t)^2+\theta'(t)^2}\,dt$

Prisjetimo se polarnih koordinata $x=r\cos\varphi$, $y=r\sin\varphi$.

Ako pomjerimo centar u x-osi za $\frac{a}{2}$ i fiksiramo r na $\frac{a}{2}$ imamo da je

$$L:\begin{cases} x=\frac{a}{2}+\frac{a}{2}\cos\varphi\\ y=\frac{a}{2}\sin\varphi\\ 0\le\varphi\le 2\pi\end{cases}$$

$$x'=-\frac{a}{2}\sin\varphi$$

$$y'=\frac{a}{2}\cos\varphi$$

$$(x')^2+(y')^2=\frac{a^2}{4}\sin^2\varphi+\frac{a^2}{4}\cos^2\varphi=\frac{a^2}{4}$$

$$\sqrt{x'^2+y'^2}=\frac{a}{2}$$

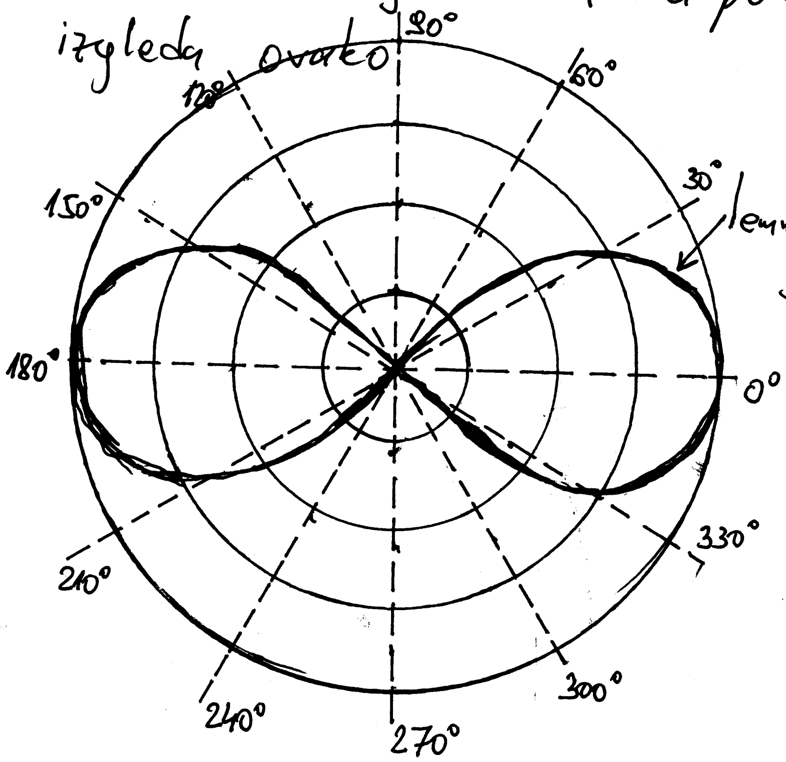
$$\int_L (x-y)\,ds=\int_0^{2\pi}\left(\frac{a}{2}+\frac{a}{2}\cos\varphi-\frac{a}{2}\sin\varphi\right)\cdot\frac{a}{2}\,d\varphi=\int_0^{2\pi}\left(\frac{a^2}{4}+\frac{a^2}{4}\cos\varphi-\frac{a^2}{4}\sin\varphi\right)d\varphi=$$

$$=\frac{a^2}{4}\varphi\Big|_0^{2\pi}+\frac{a^2}{4}\sin\varphi\Big|_0^{2\pi}+\frac{a^2}{4}\cos\varphi\Big|_0^{2\pi}=\frac{a^2}{4}\cdot 2\pi=\frac{a^2\pi}{2}$$

traženo rješenje

(#) Izračunati krivoliniski integral prve vrste $\int_c (x+y)\,dS$, ako je c desna latica lemniskate $\rho = a\sqrt{\cos 2\varphi}$.

Rj. Lemniskata $\rho = a\sqrt{\cos 2\varphi}$ u polarnom koordinatnom sistemu izgleda ovako



Data kriva je prikazana u polarnim koordinatama

$$c:\begin{cases}\rho = a\sqrt{\cos 2\varphi}\\ \varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]\end{cases}$$

Prisjetimo se,

$$\int_c (x+y)\,dS = \int_{t_1}^{t_2} (\eta(t) + \mu(t))\sqrt{(\eta'(t))^2 + (\mu'(t))^2}\,dt$$

ako je c data u obliku

$$c:\begin{cases}x = \eta(t)\\ y = \mu(t)\\ t_1 \le t \le t_2\end{cases}.$$

$$\int_{c'} (x+y)\,dS = \left|\begin{array}{l}\text{kao pomoć uvedimo polarne koordinate}\\ x = \rho\cos\varphi\\ y = \rho\sin\varphi\\ \text{za } \rho \text{ dewo uzeti}\\ \rho = a\sqrt{\cos 2\varphi}\end{array}\right.$$

Prema tome ← desna latica lemniskate

$$c':\begin{cases}x = a\cos\varphi\sqrt{\cos 2\varphi}\\ y = a\sin\varphi\sqrt{\cos 2\varphi}\\ -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}\end{cases}$$

↑ zato što posmatramo desnu stranu lemniskate

$$x' = \left(a(-\sin\varphi)\sqrt{\cos 2\varphi} + a\cos\varphi \cdot \frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}} \cdot (-\sin 2\varphi)\cdot 2\right)d\varphi = a\left(-\sin\varphi\sqrt{\cos 2\varphi} - \cos\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right)d\varphi \overset{\text{adic. teor.}}{=} \ldots = -a\frac{\sin 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$y' = \left(a\cos\varphi\sqrt{\cos 2\varphi} + a\sin\varphi \cdot \frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}}(-\sin 2\varphi)\cdot 2\right)d\varphi$$

$$= \left(a\cos\varphi\sqrt{\cos 2\varphi} - a\sin\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right)d\varphi \overset{\text{adiciona teor.}}{=} \ldots = a\frac{\cos 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$x'^2 + y'^2 = a^2\frac{\sin^2 3\varphi}{\cos 2\varphi}d\varphi^2 + a^2\frac{\cos^2 3\varphi}{\cos 2\varphi}d\varphi^2 = a^2\frac{1}{\cos 2\varphi}d\varphi^2 \quad \Bigg| =$$

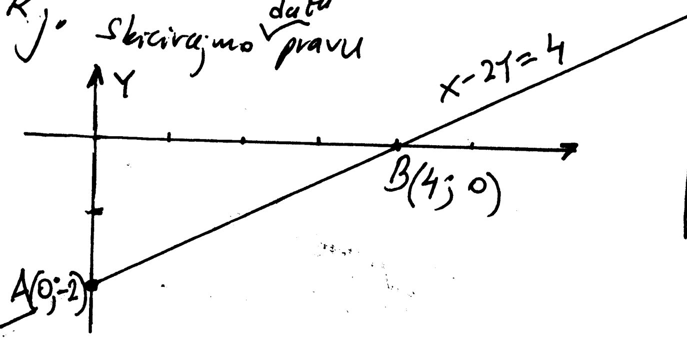
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos 2\varphi}\; a(\cos\varphi + \sin\varphi)\cdot a\frac{1}{\sqrt{\cos 2\varphi}}\,d\varphi = a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\varphi + \sin\varphi)\,d\varphi =$$

$$= a^2\left(\sin\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \cos\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\right) = a^2\sqrt{2} \quad \text{traženo rješenje.}$$

($\sin\varphi\big|$: $\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)$; $\cos\varphi\big|$: $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$)

Izračunati krivoliniski integral $\int_{AB}\frac{dl}{\sqrt{x^2+y^2}}$ po odsječku prave $x-2y=4$ od tačke $A(0;-2)$ do tačke $B(4;0)$.

Rj. Skicirajmo datu pravu



Prisjetimo se kako se računa krivolinijski integral prvog tipa, ako je kriva integracije C u ravni opisana formulom $y=\eta(x)$, $a\le x\le b$

$$\int_C f(x,y)\,dl=\int_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

I način:

$x-2y=4$

$2y=x-4$

$y=\frac{1}{2}x-2$

$y'=\frac{1}{2}$

$$\int_{AB}\frac{dl}{\sqrt{x^2+y^2}}=\int_0^4\frac{\sqrt{1+\frac{1}{4}}}{\sqrt{x^2+(\frac{1}{2}x-2)^2}}dx=\frac{\sqrt5}{2}\int_0^4\frac{dx}{\sqrt{\frac{5x^2}{4}-2x+4}}$$

$$=\frac{\sqrt5}{2}\cdot\frac{1}{\sqrt{\frac54}}\int_0^4\frac{dx}{\sqrt{x^2-\frac85x+\frac{16}{5}}}=\left|\begin{array}{l}x^2-\frac85x+\frac{16}{5}=\\ =x^2-2\cdot x\cdot\frac45+\frac{16}{25}-\frac{16}{25}+\frac{16\cdot5}{5\cdot5}\\ =(x-\frac45)^2+\frac{64}{25}\end{array}\right|$$

$$=\int_0^4\frac{d(x-\frac45)}{\sqrt{(x-\frac45)^2+\frac{64}{25}}}=\ln\left|x-\frac45+\sqrt{(x-\frac45)^2+\frac{64}{25}}\right|\Bigg|_0^4=\ln\left(\frac{16}{5}+\sqrt{\frac{16(16+4)}{25}}\right)-$$

$$-\ln\left(-\frac45+\sqrt{\frac{16+64}{25}}\right)=\ln\left(\frac{16}{5}+\frac{8\sqrt5}{5}\right)-\ln\left(-\frac45+\frac{4\sqrt5}{5}\right)$$

($16+64=16(1+4)$, $16(16+4)=4\cdot5\cdot16$)

$$=\ln\frac{\frac{16+8\sqrt5}{5}}{\frac{-4+4\sqrt5}{5}}=\ln\frac{4+2\sqrt5}{\sqrt5-1}\cdot\frac{(\sqrt5+1)}{(\sqrt5+1)}=\ln\frac{4+6\sqrt5+10}{5-1}=\ln\frac{7+3\sqrt5}{2}$$ traženo rješenje

II način

$x-2y=4$

$x=2y+4$

$\frac{\partial x}{\partial y}=2$

$$\int_{AB}\frac{dl}{\sqrt{x^2+y^2}}=\int_{-2}^0\frac{\sqrt{1+4}}{\sqrt{(2y+4)^2+y^2}}dy=\sqrt5\int_{-2}^0\frac{dy}{\sqrt{5y^2+16y+16}}=\dots$$

ZAVRŠITI ZA VJEŽBU

Neka je A tačka u kojoj prava $2x-\sqrt{5}y-1=0$ siječe y-osu, a B tačka u kojoj data prava siječe x-osu. Izračunati krivolinijski integral prve vrste $\int_c \frac{ds}{\sqrt{x^2+y^2+1}}$, ako je c odsječak date prave između tačaka A i B.

Rj.

Prisjetimo se:

Ako je kriva c data u parametarskom obliku

$$c: \begin{cases} x=\eta(t) \\ y=\mu(t) \\ t_1 \le t \le t_2 \end{cases}$$

tada se krivolinijski integral prve vrste $\int_c f(x,y)\,ds$ računa po formuli

$$\int_c f(x,y)\,ds = \int_{t_1}^{t_2} f(\eta(t),\mu(t))\sqrt{(\eta'(t))^2+(\mu'(t))^2}\,dt$$

Pronađimo presječne tačke date prave sa osama

$$2x-\sqrt{5}y-1=0$$

$$2x-\sqrt{5}y=1$$

$$\frac{x}{\frac{1}{2}}+\frac{y}{-\frac{1}{\sqrt{5}}}=0$$

Tražene tačke A i B su

$$A\left(0;-\frac{1}{\sqrt{5}}\right)$$

$$B\left(\frac{1}{2};0\right)$$

Odredimo jednačinu prave kroz tačke A i B koristeći formulu

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

$A(\overset{x_1}{0};\overset{y_1}{-\frac{1}{\sqrt{5}}})$

$B(\overset{x_2}{\frac{1}{2}};\overset{y_2}{0})$

$$\frac{x}{\frac{1}{2}}=\frac{y+\frac{1}{\sqrt{5}}}{\frac{1}{\sqrt{5}}}\quad(=t)$$

$$x=\frac{\sqrt{5}}{2}y+\frac{1}{2}$$

$$\frac{1}{\sqrt{5}}x=\frac{1}{2}\left(y+\frac{1}{\sqrt{5}}\right)\quad/\cdot\sqrt{5}$$

$$\frac{\sqrt{5}}{2}y=x-\frac{1}{2}\quad/\cdot\frac{2}{\sqrt{5}}$$

$$x=\frac{1}{2}t$$

$$y+\frac{1}{\sqrt{5}}=\frac{1}{\sqrt{5}}t$$

$\Rightarrow$ Parametarski oblik duži c je

$$c:\begin{cases}x=\frac{1}{2}t\\ y=\frac{1}{\sqrt{5}}t-\frac{1}{\sqrt{5}}\\ 0\le t\le 1\end{cases}\Rightarrow$$

$$\dot{x}=\frac{1}{2}$$

$$\dot{y}=\frac{1}{\sqrt{5}}$$

$$\Rightarrow \dot{x}^2+\dot{y}^2=\frac{1}{4}+\frac{1}{5}=$$

$$=\frac{5+4}{20}=\frac{9}{20}$$

$$\sqrt{20}=\sqrt{5\cdot 4}=2\sqrt{5}$$

$$x^2+y^2+1=\left(\frac{1}{2}t\right)^2+\left(\frac{1}{\sqrt{5}}t-\frac{1}{\sqrt{5}}\right)^2+1\overset{\text{ZA VJEŽBU}}{=}\ldots=\frac{9}{20}t^2-\frac{2}{5}t+\frac{6}{5}$$

$$\sqrt{x^2+y^2+1}=\sqrt{\frac{9}{20}}\cdot\sqrt{t^2-\frac{8t}{9}+\frac{8}{3}}=\frac{3}{2\sqrt{5}}\sqrt{\left(t-\frac{4}{9}\right)^2+\frac{200}{81}}$$

$$\int_c\frac{ds}{\sqrt{x^2+y^2+1}}=\frac{3}{2\sqrt{5}}\cdot\frac{2\sqrt{5}}{3}\int_0^1\frac{dt}{\sqrt{\left(t-\frac{4}{9}\right)^2+\frac{200}{81}}}=\left|\begin{array}{ll}t-\frac{4}{9}=\frac{10\sqrt{2}}{9}s & t=0\Rightarrow\ldots\\ \left(t-\frac{4}{9}\right)^2=\frac{200}{81}s^2 & t=1\Rightarrow\ldots\end{array}\right|$$

$$\overset{\text{ZA VJEŽBU}}{=}\ldots=\ln\left(\frac{3\sqrt{6}}{5}+\frac{2}{5}\right)$$

(7) Izračunati krivoliniske integrale

a) $\oint_{-l} 2x\,dx-(x+2y)\,dy$ i b) $\oint_{+l} y\cos x\,dx+\sin x\,dy$

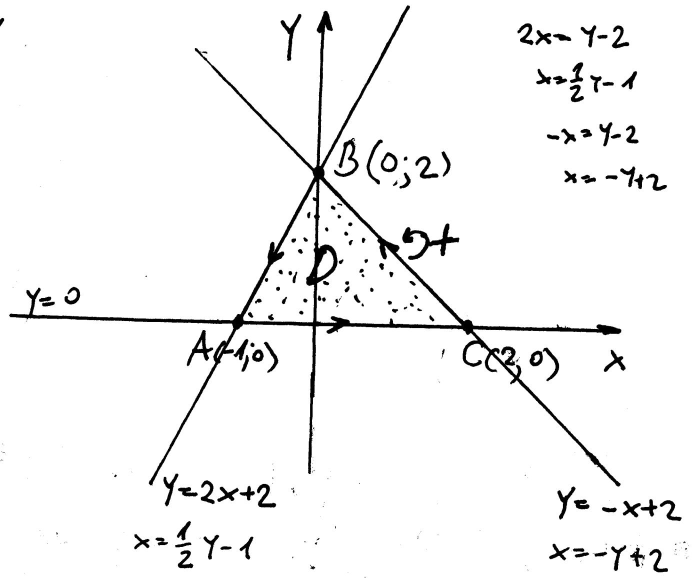
po krivoj l, gdje je l trougao čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

Rj.

$\int_c P(x,y)\,dx + Q(x,y)\,dy$ je krivoliniski integral druge vrste.

Ako je kriva c data u obliku $y=\eta(x)$, $c_1 \le x \le c_2$ dati integral se računa po formuli

$$\int_{c_1}^{c_2}\left(P(x,\eta(x)) + Q(x,\eta(x))\,\eta'(x)\right)dx$$



Skicirajmo tačke u xOy ravni

prava koja prolazi kroz tačke A i B je

$$\frac{x}{-1}+\frac{y}{2}=1 \quad /\cdot 2$$

$$-2x+y=1$$

$$y=2x+2 \quad \Rightarrow \quad y'=2$$

prava koja prolazi kroz tačke B i C

$$\frac{x}{2}+\frac{y}{2}=1 \quad /\cdot 2$$

$$x+y=2$$

$$y=-x+2 \quad \Rightarrow \quad y'=-1$$

a)
$$\oint_{-l} 2x\,dx-(x+2y)\,dy = \int_{AB} 2x\,dx-(x+2y)\,dy + \int_{BC} 2x\,dx-(x+2y)\,dy + \int_{CA} 2x\,dx-(x+2y)\,dy$$

AB po pravoj $y=2x+2$; BC po pravoj $y=-x+2$; CA po pravoj $y=0$

$$\int_{AB} 2x\,dx-(x+2y)\,dy = \int_{-1}^{0}\Big[2x-\underbrace{(x+2(2x+2))2}_{\substack{(x+4x+4)\cdot 2\\ 10x+8}}\Big]dx = \int_{-1}^{0}(-8x-8)\,dx = -8\cdot\frac{1}{2}x^2\Big|_{-1}^{0} - 8x\Big|_{-1}^{0}$$

$$= (-4)(-1)-8 = -4$$

$$\int_{BC} 2x\,dx-(x+2y)\,dy=\int_0^2 \left[2x-(x+2(-x+2))(-1)\right]dx=\int_0^2 (x+4)\,dx=\frac{1}{2}x^2\Big|_0^2+$$

BC po pravoj $y=-x+2$

$(x-2x+4)(-1)$

$x-4$

$$+4x\Big|_0^2=2+8=10$$

$$\int_{CA} 2x\,dx-(x+2y)\,dy=\int_2^{-1}\left[2x-(x+2(0))0\right]dx=\int_2^{-1}2x\,dx=2\cdot\frac{1}{2}x^2\Big|_2^{-1}=1-4=-3$$

CA po pravoj $y=0$

$$\oint_{-\ell} 2x\,dx-(x+2y)\,dy=-4+10-3=3$$ traženo rješenje

b) Možemo upotrebiti Greenovu formulu

$$\int_C P(x,y)\,dx+Q(x,y)\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$ gdje je D oblast ograničena konturom C

$$\oint_{+\ell} y\cos x\,dx+\sin x\,dy=\left|\begin{matrix} Q(x,y)=\sin x & P(x,y)=y\cos x \\ \frac{\partial Q}{\partial x}=\cos x & \frac{\partial P}{\partial y}=\cos x \\ D\text{-vidi sliku (tačkasti dio sa slike)} & \end{matrix}\right|=$$

$$=\iint_D(\cos x-\cos x)\,dx\,dy=\iint_D 0\,dx\,dy=0$$ traženo rješenje

(#) Date su tačke $A(3; -6; 0)$ i $B(-2; 4; 5)$. Izračunati krivolinijski integral $I = \int_c xy^2dx + yz^2dy - zx^2dz$ gdje je c:

a) duž koja spaja tačke O i B (O koordinatni početak)

b) kriva od A do B kruga zadan jednačinama $x^2+y^2+z^2=45$, $2x+y=0$.

Rj.

$$I = \int_c xy^2dx + yz^2dy + zx^2dz$$

Ovo je krivolinijski integral druge vrste. Prisjetimo se: Ako je c kriva u prostoru opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$, $z=\theta(t)$ gdje je $t_1 \le t \le t_2$ tada

$$\int_c P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \int_{t_1}^{t_2} P(\mu(t),\eta(t),\theta(t))\mu'(t)\,dt +$$

$$+ \int_{t_1}^{t_2} Q(\mu(t),\eta(t),\theta(t))\eta'(t)\,dt + \int_{t_1}^{t_2} R(\mu(t),\eta(t),\theta(t))\theta'(t)\,dt$$

Da bi smo opisali duž $\overline{OB}$ u prostoru prvo postavimo pravu kroz ove dvije tačke.

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke $M_1(x_1,y_1,z_1)$ i $M_2(x_2,y_2,z_2)$

$O(0,0,0)$

$B(-2,4,5)$

$$\frac{x}{-2} = \frac{y}{4} = \frac{z}{5} \quad (=t)$$

$x=-2t$

$y=4t$

$z=5t$

Naše c je sada oblika

$$c: \begin{cases} x=-2t, \ y=4t, \ z=5t \\ 0 < t < 1 \end{cases}$$

$$I = \int_c xy^2dx + yz^2dy - zx^2dz = \int_0^1 \left((-2t)\,16t^2\cdot(-2) + 4t\cdot 25t^2\cdot 4 - 5t\cdot 4t^2\cdot 5\right)dt =$$

$$= \int_0^1 (64t^3 + 400t^3 - 100t^3)\,dt = 364\int_0^1 t^3 dt = \frac{364}{4} = 91$$ traženo rješenje

b) Dat je krug u prostoru zadan jednačinama

$x^2+y^2+z^2=45$, $2x+y=0$

↑ krug ↑ ravan

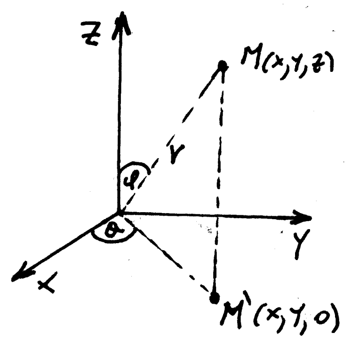
Da bi smo naš krug opisali u parametarskom obliku, veliku pomoć će odigrati sferne koordinate.

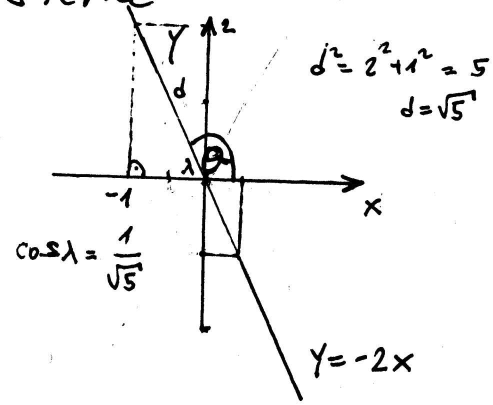
Sferne koordinate

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$





Da bi smo krug u prostoru opisali parametarski potrebno je u sfernim koordinatama fiksirati r i θ. U našem slučaju ugao θ nije moguće svesti na lijep oblik.
Pristupimi parametarizaciji kruga na drugi način:

$2x+y=0 \Rightarrow y=-2x$

$x^2+y^2+z^2=45 \Rightarrow z^2 = 45-x^2-y^2$

$$\Rightarrow \quad c: \begin{cases} x = t \\ y = -2t \\ z = \sqrt{45-t^2-4t^2} = \sqrt{45-5t^2} \\ 3 \le t \le -2 \end{cases}$$

$$dx = dt, \quad dy = -2dt, \quad dz = \frac{1}{2}(45-5t^2)^{-\frac{1}{2}}\cdot(-10t) = -\frac{5t}{\sqrt{45-5t^2}}\,dt$$

$$I = \int_c xy^2dx + yz^2dy - zx^2dz = \int_3^{-2}\left(t\cdot 4t^2 + (-2t)(45-5t^2)\cdot(-2) - \sqrt{45-5t^2}\cdot t^2\frac{(-5t)}{\sqrt{45-5t^2}}\right)dt$$

$$= \int_3^{-2}(4t^3 + 180t - 20t^3 + 5t^3)\,dt = \int_3^{-2}(-11t^3 + 180t)\,dt$$

$$= -11\cdot\frac{1}{4}t^4\Big|_3^{-2} + 180\cdot\frac{1}{2}t^2\Big|_3^{-2} = -\frac{11}{4}\cdot(-65) + 90\cdot(-5) = \frac{715-1800}{4} = \frac{-1085}{4}$$

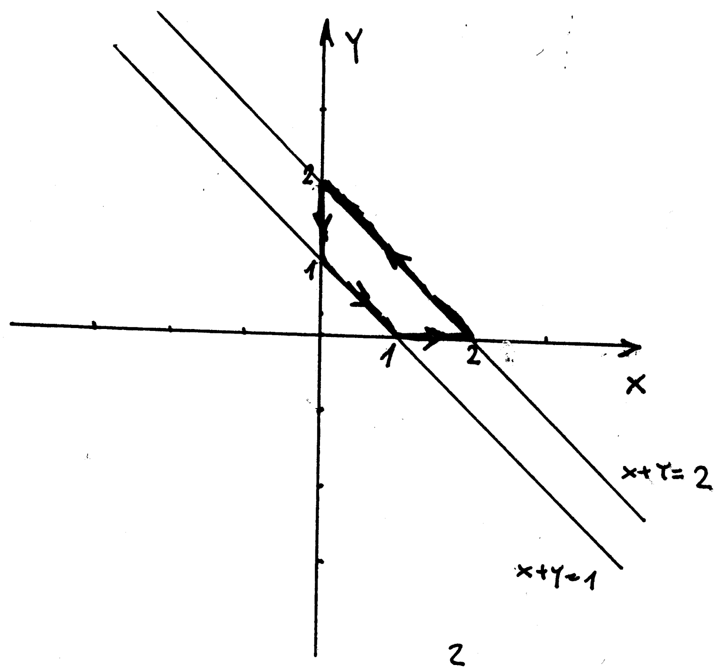
$$= -271\frac{1}{4}$$ traženo rješenje

(#) Izračunati krivoliniski integral $I=\int\limits_c (x^2+y^2)\,dx + x^2y\,dy$

gdje je c kontura trapeza koja obrazuju prave

$x=0$, $y=0$, $x+y=1$, $x+y=2$.

Rj.



Ako je $c:\ y=\eta(x),\ a\le x\le b$

$$\int\limits_c P(x,y)\,dx+Q(x,y)\,dy = \int_a^b [P(x,\eta(x)) + Q(x,\eta(x))\cdot\eta'(x)]\,dx$$

U našem slučaju postoje 4 krive

$C_1:\ y=0,\ 1\le x\le 2$

$C_1:\ y=-x+2,\ 2\ge x\ge 0$

$C_3:\ x=0,\ 2\ge y\ge 1$

$C_4:\ y=-x+1,\ 0\le x\le 1$

$I=I_1+I_2+I_3+I_4$, $\quad I_1=\int_1^2 (x^2+x^2\cdot 0)\,dx=\int_1^2 x^2\,dx=\frac{1}{3}x^3\Big|_1^2=\frac{1}{3}(8-1)=\frac{7}{3}$

$$I_2=\int_2^0 \left(x^2+(-x+2)^2+x^2(-x+2)\cdot(-1)\right)dx=\int_2^0 (x^2+x^2-4x+4+x^3-2x^2)\,dx=$$

$$=\int_2^0 (x^3-4x+4)\,dx=\frac{1}{4}x^4\Big|_2^0-4\cdot\frac{1}{2}x^2\Big|_2^0+4x\Big|_2^0=-4+8-8=-4$$

$$I_3=\int_2^1 (y^2\cdot 0+0\cdot y)\,dy=0$$

$$I_4=\int_0^1 \left(x^2+(-x+1)^2+x^2(-x+1)\cdot(-1)\right)dx=\int_0^1 (x^2+x^2-2x+1+x^3-x^2)\,dx=$$

$$=\int_0^1 (x^3+x^2-2x+1)\,dx=\frac{1}{4}x^4\Big|_0^1+\frac{1}{3}x^3\Big|_0^1-2\cdot\frac{1}{2}x^2\Big|_0^1+x\Big|_0^1=\frac{1}{4}+\frac{1}{3}-1+1=\frac{7}{12}$$

$I=I_1+I_2+I_3+I_4=\frac{7}{3}+(-4)+\frac{7}{12}=\frac{28-48+7}{12}=-\frac{13}{12}$ vrijednost krivoliniskog integrala

II način: Greenova formula ...

Izračunati krivolinijski integral

$$I=\oint_C z\,dz$$

duž krive koja nastaje kao presjek cilindra $\frac{(x-\frac{a}{2})^2}{\frac{a^2}{2}}+\frac{(y-\frac{b}{2})^2}{\frac{b^2}{2}}=1$ i paraboloida $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ orjentisana u pozitivnom smjeru $(a\geq b>0)$.

Rj. Prisjetimo se

Ako je kriva $C:\begin{cases}x=\mu(t)\\ y=\eta(t)\\ z=\theta(t)\\ t_1\leq t\leq t_2\end{cases}$ data u parametarskom obliku, tada

$$\int_C P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz=\int_{t_1}^{t_2}\Big(P(\mu(t),\eta(t),\theta(t))\mu'(t)+Q(\mu(t),\eta(t),\theta(t))\eta'(t)$$
$$+R(\mu(t),\eta(t),\theta(t))\theta'(t)\Big)\,dt$$

Da bi izračunali dati integral trebamo parametrizirati datu krivu. Uvrstimo smjene za x i y t.d. $\frac{(x-\frac{a}{2})^2}{\frac{a^2}{2}}+\frac{(y-\frac{b}{2})^2}{\frac{b^2}{2}}=1$. (*)

Za x i y mogu nam pomoći poopštene polarne koordinate (gdje je r fiksirano)

$$x=\frac{a}{2}+\frac{a}{\sqrt{2}}\cos\varphi \Rightarrow \left(x-\frac{a}{2}\right)^2=\frac{a^2}{2}\cos^2\varphi$$
$$y=\frac{b}{2}+\frac{b}{\sqrt{2}}\sin\varphi \Rightarrow \left(y-\frac{b}{2}\right)^2=\frac{b^2}{2}\sin^2\varphi$$

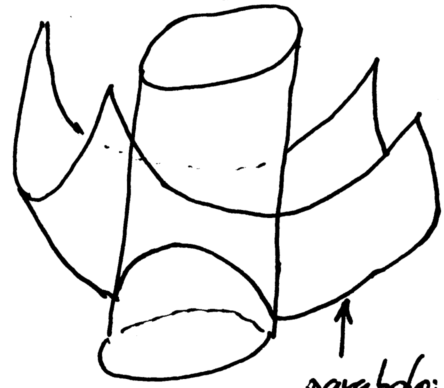
$\Rightarrow$ vrijedi (*) za $\varphi\in[0,2\pi)$

Sada je

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{\left(\frac{a}{2}+\frac{a}{\sqrt{2}}\cos\varphi\right)^2}{a^2}+\frac{\left(\frac{b}{2}+\frac{b}{\sqrt{2}}\sin\varphi\right)^2}{b^2}=\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\cos\varphi\right)^2+\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\sin\varphi\right)^2=$$

$$= \frac{1}{4} + \frac{1}{\sqrt{2}}\cos\varphi + \frac{1}{2}\cos^2\varphi + \frac{1}{4} + \frac{1}{\sqrt{2}}\sin\varphi + \frac{1}{2}\sin^2\varphi =$$

$$= 1 + \frac{1}{\sqrt{2}}\cos\varphi + \frac{1}{\sqrt{2}}\sin\varphi.$$



Prema tome imamo

$$z = 1 + \frac{1}{\sqrt{2}}\cos\varphi + \frac{1}{\sqrt{2}}\sin\varphi$$

$$dz = -\frac{1}{\sqrt{2}}\sin\varphi + \frac{1}{\sqrt{2}}\cos\varphi$$

$$\oint_c z\,dz = \int_0^{2\pi}\left(1 + \frac{1}{\sqrt{2}}\cos\varphi + \frac{1}{\sqrt{2}}\sin\varphi\right)\left(-\frac{1}{\sqrt{2}}\sin\varphi + \frac{1}{\sqrt{2}}\cos\varphi\right)d\varphi =$$

$$= \int_0^{2\pi}\Big(-\frac{1}{\sqrt{2}}\sin\varphi + \frac{1}{\sqrt{2}}\cos\varphi \underline{- \frac{1}{2}\sin\varphi\cos\varphi} + \underbrace{\frac{1}{2}\cos^2\varphi - \frac{1}{2}\sin^2\varphi}_{=\frac{1}{2}(\cos^2\varphi - \sin^2\varphi)} \underline{+ \frac{1}{2}\sin\varphi\cos\varphi}\Big)d\varphi$$

$$= -\frac{1}{\sqrt{2}}\int_0^{2\pi}\sin\varphi\,d\varphi + \frac{1}{\sqrt{2}}\int_0^{2\pi}\cos\varphi\,d\varphi + \frac{1}{2}\int_0^{2\pi}\cos 2\varphi\,d\varphi =$$

$$= -\frac{1}{\sqrt{2}}(-\cos\varphi)\Big|_0^{2\pi} + \frac{1}{\sqrt{2}}\sin\varphi\Big|_0^{2\pi} + \frac{1}{2}\cdot\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi} =$$

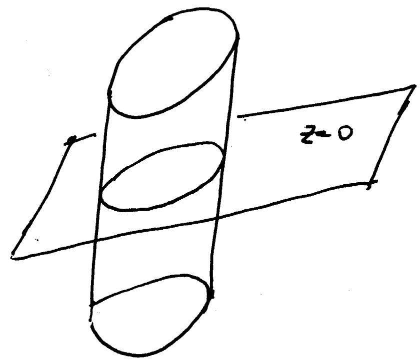
$$= \frac{1}{\sqrt{2}}(1-1) + 0 + 0 = 0$$

(#) Izračunati krivolinijski integral

$$I = \oint_C y\,dx + x^2\,dy$$

duž krive koja nastaje kao presjek ravni $z=0$ i cilindra $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}$ orjentisana u pozitivnom smjeru $(a \geq b > 0)$.

Rj. Za rješenje zadatka nije nam bitno gdje se cilindar nalazi u prostoru



$z=0$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}$$

$$\frac{1}{a^2}(x^2 - ax) + \frac{1}{b^2}(y^2 - by) = 0$$

$$\frac{1}{a^2}\left(x^2 - 2\cdot x\cdot\frac{a}{2} + \frac{a^2}{4} - \frac{a^2}{4}\right) + \frac{1}{b^2}\left(y^2 - 2\cdot y\cdot\frac{b}{2} + \frac{b^2}{4} - \frac{b^2}{4}\right) = 0$$

$$\frac{1}{a^2}\left(x - \frac{a}{2}\right)^2 - \frac{1}{4} + \frac{1}{b^2}\left(y - \frac{b}{2}\right)^2 - \frac{1}{4} = 0$$

$$\frac{\left(x - \frac{a}{2}\right)^2}{a^2} + \frac{\left(y - \frac{b}{2}\right)^2}{b^2} = \frac{1}{2} \quad /\cdot 2$$

$$\frac{\left(x - \frac{a}{2}\right)^2}{\frac{a^2}{2}} + \frac{\left(y - \frac{b}{2}\right)^2}{\frac{b^2}{2}} = 1$$

ovo je elipsa

Elipsu ćemo parametrizirati pomoću poopćenih polarnih koordinata

$$x = \frac{a}{2} + \frac{a}{\sqrt{2}} \cos\varphi \qquad dx = -\frac{a}{\sqrt{2}} \sin\varphi \, d\varphi$$

$$y = \frac{b}{2} + \frac{b}{\sqrt{2}} \sin\varphi \qquad dy = \frac{b}{\sqrt{2}} \cos\varphi \, d\varphi$$

$$0 \le \varphi < 2\pi$$

Sad nije teško izračunati dati krivolinijski integral

$$I = \oint_C y\,dx + x^2 dy = \int_0^{2\pi} \left[\left(\frac{b}{2} + \frac{b}{\sqrt{2}} \sin\varphi\right) \frac{-a}{\sqrt{2}} \sin\varphi + \left(\frac{a}{2} + \frac{a}{\sqrt{2}} \cos\varphi\right)^2 \cdot \frac{b}{\sqrt{2}} \cos\varphi \right] d\varphi$$

$$= \frac{-a}{\sqrt{2}} \int_0^{2\pi} \left(\frac{b}{2} + \frac{b}{\sqrt{2}} \sin\varphi\right) \sin\varphi \, d\varphi + \frac{b}{\sqrt{2}} \int_0^{2\pi} \left(\frac{a^2}{4} + \frac{a^2}{\sqrt{2}} \cos\varphi + \frac{a^2}{2} \cos^2\varphi\right) \cos\varphi \, d\varphi$$

$$= \frac{-ab}{2\sqrt{2}} \int_0^{2\pi} \sin\varphi \, d\varphi - \frac{ab}{2} \int_0^{2\pi} \underbrace{\sin^2\varphi}_{\frac{1}{2}(1-\cos 2\varphi)} d\varphi + \frac{a^2 b}{4\sqrt{2}} \int_0^{2\pi} \cos\varphi \, d\varphi + \frac{a^2 b}{2} \int_0^{2\pi} \underbrace{\cos^2\varphi}_{\frac{1}{2}(1+\cos 2\varphi)} d\varphi + \frac{a^2 b}{2\sqrt{2}} \int_0^{2\pi} \cos^3\varphi \, d\varphi$$

$$= \frac{-ab}{2\sqrt{2}} \underbrace{\cos\varphi \Big|_0^{2\pi}}_{=1-1=0} - \frac{ab}{2} \cdot \frac{1}{2} \left(\varphi \Big|_0^{2\pi} - \underbrace{\frac{1}{2} \sin 2\varphi \Big|_0^{2\pi}}_{=0} \right) + 0 + \frac{a^2 b}{2} \cdot \frac{1}{2} \left(\varphi \Big|_0^{2\pi} + \underbrace{\frac{1}{2} \sin 2\varphi \Big|_0^{4\pi}}_{=0} \right)$$

$$+ \frac{a^2 b}{2\sqrt{2}} \int_0^{2\pi} \cos^2\varphi \cos\varphi \, d\varphi =$$

$$= -\frac{ab\pi}{2} + \frac{a^2 b \pi}{2} + \frac{a^2 b}{2\sqrt{2}} \underbrace{\int_0^{2\pi} (1 - \sin^2\varphi) \, d(\sin\varphi)}_{=0}$$

$$= \frac{\pi ab(-1+a)}{2} = \frac{ab\pi}{2}(a-1)$$

traženo rješenje

$$1 = \sin^2\varphi + \cos^2\varphi$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$

$$1 - \cos 2\varphi = 2\sin^2\varphi$$

$$\sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi)$$

$$1 + \cos 2\varphi = 2\cos^2\varphi$$

II način: Greenova formula ...

(#) Izračunati integral $I=\oint_C y^2\,dx$

po krivoj koja nastaje kao presjek kugle i valjka

$$x^2+y^2+z^2=R^2, \quad x^2+y^2=Rx.$$

Rj.:

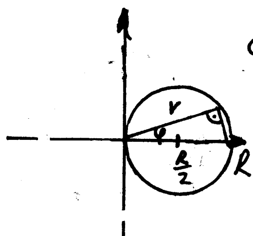
$$C: \begin{cases} x^2+y^2+z^2=R^2 \\ x^2+y^2=Rx \end{cases}$$

Posmatrajmo xOy ravan.
Prvo napišimo krug $x^2+y^2=Rx$ u parametarskom obliku.

$$x^2+y^2=Rx$$

$$x^2-2\cdot x\cdot\frac{R}{2}+\frac{R^2}{4}-\frac{R^2}{4}+y^2=0$$

$$\left(x-\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2$$



$\cos\varphi=\frac{r}{R}$

$\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Prisjetimo se polarnih koordinata
$x=r\cos\varphi$
$y=r\sin\varphi$

U našem slučaju za krug $x^2+y^2=Rx$ za r ćemo uzeti $r=R\cos\varphi$
Parametarski oblik kruga $x^2+y^2=Rx$ je $x=R\cos\varphi\cos\varphi=R\cos^2\varphi$
$y=R\cos\varphi\sin\varphi.$

Uvrstimo ove vrijednosti u kuglu

$$x^2+y^2+z^2=R^2$$

$$R^2\cos^2\varphi\cos^2\varphi+R^2\cos^2\varphi\sin^2\varphi+z^2=R^2$$

$$R^2\cos^2\varphi+z^2=R^2$$

$$z^2=R^2-R^2\cos^2\varphi$$

$$z^2=R^2(1-\cos^2\varphi)$$

$$z^2=R^2\sin^2\varphi$$

Parametarski oblik date krive je:
$x=R\cos^2\varphi$
$y=R\cos\varphi\sin\varphi$
$z=R\sin\varphi$
$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

$$I=\oint_C y^2\,dx=\left|\begin{array}{l} x=R\cos^2\varphi \\ dx=R\,2\cos\varphi(-\sin\varphi)\,d\varphi \\ y=R\cos\varphi\sin\varphi \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{array}\right|=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}R^2\cos^2\varphi\sin^2\varphi\cdot(-2)R\sin\varphi\cos\varphi\,d\varphi$$

$$=(-2)R^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3\varphi\cos^3\varphi\,d\varphi=(-2)R^3\cdot\frac{1}{8}\int_{-\pi/2}^{\pi/2}(2\sin\varphi\cos\varphi)^3\,d\varphi=-\frac{1}{4}R^3\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin 2\varphi)^3\,d(2\varphi)$$

$$= -\frac{1}{8}R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\varphi \cdot \sin 2\varphi \, d(2\varphi) = -\frac{1}{8}R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos^2 2\varphi) \sin 2\varphi \, d(2\varphi) =$$

$$= +\frac{1}{8}R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos^2 2\varphi) \cdot d(\cos 2\varphi) = \frac{1}{8}R^3 \Big(\underbrace{\cos 2\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{-1+1} - \underbrace{\frac{1}{3}\cos^3 2\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{-1+1} \Big) = 0$$

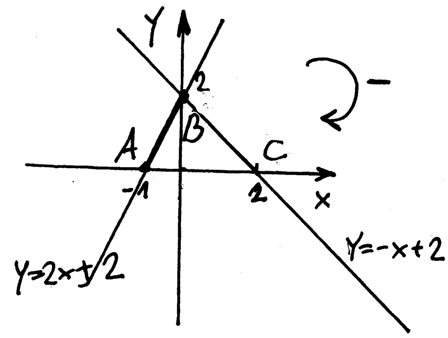
#) Izračunati krivolinijske integrale

a) $\oint\limits_{-\ell} 2x\,dx - (x+2y)\,dy$

b) $\oint\limits_{+\ell} y\cos x\,dx + \sin x\,dy$

gdje je ℓ kontura trougla čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

Rj. a)
Nacrtajmo trougao ΔABC.



Provucimo pravu kroz tačke $B(\overset{x_1}{0};\overset{y_1}{2})$ i $C(\overset{x_2}{2};\overset{y_2}{0})$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

$$\frac{x}{2}=\frac{y-2}{-2} \quad /\cdot 2$$

$$x=-y+2$$

$$y=-x+2$$

Provucimo pravu kroz $A(\overset{x_1}{-1};\overset{y_1}{0})$ i $B(\overset{x_2}{0};\overset{y_2}{2})$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad \frac{x+1}{1}=\frac{y}{2}$$

$$y=2x+2$$

$$\oint\limits_{-\ell} 2x\,dx-(x+2y)\,dy=\int\limits_{B(0;2)}^{C(2;0)} 2x\,dx-(x+2y)\,dy+\int\limits_{C(2;0)}^{A(-1;0)} 2x\,dx-(x+2y)\,dy+\int\limits_{A(-1;0)}^{B(0;2)} 2x\,dx-(x+2y)\,dy$$

$$\int\limits_{(0;2)}^{(2;0)} 2x\,dx-(x+2y)\,dy=\left|\begin{matrix} y=-x+2\\ dy=-dx\end{matrix}\right|=\int\limits_{(0;2)}^{(2;0)}\left[2x-(x+2(-x+2))(-1)\right]dx=$$

$$=\int\limits_{(0;2)}^{(2;0)}\left[2x+x-2x+4\right]dx=\int\limits_{(0;2)}^{(2;0)}(x+4)\,dx=\left(\frac{1}{2}x^2+4x\right)\Big|_0^2=2+8=10$$

$$\int\limits_{C(2;0)}^{A(-1;0)} 2x\,dx-(x+2y)\,dy=\left|\begin{matrix} y=0\\ dy=0\end{matrix}\right|=\int\limits_{2}^{-1}2x\,dx=2\cdot\frac{1}{2}x^2\Big|_2^{-1}=(1-4)=-3$$

$$\int\limits_{A(-1;0)}^{B(0;2)} 2x\,dx-(x+2y)\,dy=\left|\begin{matrix} y=2x+2\\ dy=2\,dx\end{matrix}\right|=\int\limits_{-1}^{0}\left[2x-(x+2(2x+2))\,2\right]dx=$$

$$= \int_{-1}^{0} (2x - 2x - 8x - 8)\, dx = (-8) \int_{-1}^{0} (x+1)\, dx = (-8) \Big[\underbrace{\tfrac{1}{2} x^2 \Big|_{-1}^{0}}_{0-1} + \underbrace{x \Big|_{-1}^{0}}_{0+1} \Big] =$$

$$= (-8)\left(-\frac{1}{2} + 1\right) = -4$$

Prema tome $\oint_{\triangle ABC} 2x\,dx - (x+2y)\,dy = 10 - 3 - 4 = 3$

b) $$\oint_{+\ell} y\cos x\,dx + \sin x\,dy = \int_{AC} y\cos x\,dx + \sin x\,dy + \int_{CB} y\cos x\,dx + \sin x\,dy$$
$$+ \int_{BA} y\cos x\,dx + \sin x\,dy$$

$$\int_{A(-1;0)}^{C(2;0)} y\cos x\,dx + \sin x\,dy = \begin{vmatrix} y = 0 \\ dy = 0 \end{vmatrix} = \int_{-1}^{2} 0\,dx = 0$$

$$\int_{C(2;0)}^{B(0;2)} y\cos x\,dx + \sin x\,dy = \begin{vmatrix} y = -x+2 \\ dy = -dx \end{vmatrix} = \int_{2}^{0} \left[(-x+2)\cos x - \sin x\right] dx$$

$$= \begin{vmatrix} u = -x+2 & dv = \cos x \\ du = -1 & v = \sin x \end{vmatrix} = (-x+2)\sin x \Big|_{2}^{0} + \int_{2}^{0} \sin x\,dx - \int_{2}^{0} \sin x\,dx = 0$$

$$\int_{B(0;2)}^{A(-1;0)} y\cos x\,dx + \sin x\,dy = \begin{vmatrix} y = 2x+2 \\ dy = 2dx \end{vmatrix} = \int_{0}^{-1} \left[(2x+2)\cos x + 2\sin x\right] dx =$$

$$= 2\int_{0}^{-1} \left[(x+1)\cos x + \sin x\right] dx = \begin{vmatrix} u = x+1 & dv = \cos x \\ du = dx & v = \sin x \end{vmatrix} = 2(x+1)\sin x \Big|_{0}^{-1} - 2\int_{0}^{-1} \sin x\,dx$$

$$+ 2\int_{0}^{-1} \sin x\,dx = 0$$ Prema tome

$$\oint_{+\ell} y\cos x\,dx + \sin x\,dy = 0$$

(#) Izračunati krivolinijski integral druge vrste

$$I = \oint_C x\,dy + x\,dz$$

gdje je C kriva koja nastaje presjekom cilindrične površi $x^2+y^2=2x$ i ravni $z=x$ pozitivno orjentisana ako se posmatra iz tačke $(0;0;1)$.

Rj.

$$C: \begin{cases} x^2+y^2=2x \\ z=x \end{cases}$$

$$x^2+y^2=2x$$
$$x^2-2\cdot x+1+y^2=1$$
$$(x-1)^2+y^2=1$$

krug sa centrom $C(1,0)$ poluprečnika 1

$$x-1=\cos\varphi$$
$$y=\sin\varphi$$

$$\Rightarrow \quad x^2+y^2=2x: \begin{cases} x=1+\cos\varphi \\ y=\sin\varphi \end{cases}$$

Sad nije teško parametrizirati datu krivu C.

$$C: \begin{cases} x=1+\cos\varphi \\ y=\sin\varphi \\ z=1+\cos\varphi \\ 0\le\varphi\le 2\pi \end{cases}$$

$$dx=-\sin\varphi\,d\varphi$$
$$dy=\cos\varphi\,d\varphi$$
$$dz=-\sin\varphi\,d\varphi$$

$$I=\oint_C x\,dy+x\,dz=\int_0^{2\pi}\left[(1+\cos\varphi)\cos\varphi+(1+\cos\varphi)(-\sin\varphi)\right]d\varphi=$$

$$=\int_0^{2\pi}\cos\varphi\,d\varphi+\int_0^{2\pi}\cos^2\pi\,d\varphi-\int_0^{2\pi}\sin\varphi\,d\varphi-\int_0^{2\pi}\sin\varphi\cos\varphi\,d\varphi=$$

$= \underset{\ldots}{\text{ZA VJEŽBU}} = 0+\pi-0-0=\pi$ traženo rješenje

(#) Izračunati krivolinijski integral druge vrste

$$I = \oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$$ gdje je C krug

$x^2+y^2+z^2=a^2$ $(a>0)$, $y = x\,tg\alpha$, $(0<\alpha<\frac{\pi}{2})$ uzet u smjeru suprotnom kretanju kazaljke na satu ako se posmatra sa pozitivnog dijela x-ose.

Rj.

$$C: \begin{cases} x^2+y^2+z^2=a^2, (a>0) \\ y = x\,tg\alpha \quad (0<\alpha<\frac{\pi}{2}) \end{cases}$$

Parametrizujmo krivu C. Kako je $tg\alpha = \frac{\sin\alpha}{\cos\alpha}$ to možemo npr. uzeti $x = a\cos\alpha\sin\varphi$. Tada,

$$y = x\,tg\alpha = a\cos\alpha\sin\varphi\frac{\sin\alpha}{\cos\alpha} = a\sin\alpha\sin\varphi$$

Dalje iz $x^2+y^2+z^2=a^2$ imamo

$$(a\cos\alpha\sin\varphi)^2 + (a\sin\alpha\sin\varphi)^2 + z^2 = a^2$$

$$a^2\underbrace{(\cos^2\alpha+\sin^2\alpha)}_{=1}\sin^2\varphi + z^2 = a^2$$

$$z^2 = a^2 - a^2\sin^2\varphi$$

$$z^2 = a^2(1-\sin^2\varphi) \Rightarrow z = a\cos\varphi$$

Dati krug C ima sljedeću parametrizaciju

$$C: \begin{cases} x = a\cos\alpha\sin\varphi \\ y = a\sin\alpha\sin\varphi \\ z = a\cos\varphi \\ 0 \le \varphi \le 2\pi \end{cases}$$

$dx = a\cos\alpha\cos\varphi\,d\varphi$

$dy = a\sin\alpha\cos\varphi\,d\varphi$

$dz = -a\sin\varphi$

$$\oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz =$$

$$= \int_0^{2\pi} \Big[(a\sin\alpha\sin\varphi - a\cos\varphi)\,a\cos\alpha\cos\varphi + (a\cos\varphi - a\cos\alpha\sin\varphi)\,a\sin\alpha\cos\varphi$$
$$+ (a\cos\alpha\sin\varphi - a\sin\alpha\sin\varphi)(-a)\sin\varphi \Big]\,d\varphi$$

$$= \underline{a^2\sin\alpha\cos\alpha \int_0^{2\pi} \sin\varphi\cos\varphi\,d\varphi} - a^2\cos\alpha \int_0^{2\pi} \cos^2\varphi\,d\varphi + a^2\sin\alpha \int_0^{2\pi} \cos^2\varphi\,d\varphi$$

$$\underline{- a^2\sin\alpha\cos\alpha \int_0^{2\pi} \sin\varphi\cos\varphi\,d\varphi} - a^2\cos\alpha \int_0^{2\pi} \sin^2\varphi\,d\varphi + a^2\sin\alpha \int_0^{2\pi} \sin^2\varphi\,d\varphi$$

$$= -a^2\cos\alpha \int_0^{2\pi} \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=1}\,d\varphi + a^2\sin\alpha \int_0^{2\pi} \underbrace{(\sin^2\varphi + \cos^2\varphi)}_{=1}\,d\varphi$$

$$= 2\pi a^2(\sin\alpha - \cos\alpha) = 2a^2(\sin\alpha - \cos\alpha)\pi$$

traženo rješenje

(#) Izračunati vrijednost krivoliniskog integrala

$$\oint_c y\,dx + z\,dy + x\,dz$$

duž zatvorene krive c koja je dobijena kao presjek sljedećih površina

$$x^2+y^2=r^2 \quad i \quad x^2 = rz \quad (r>0)$$

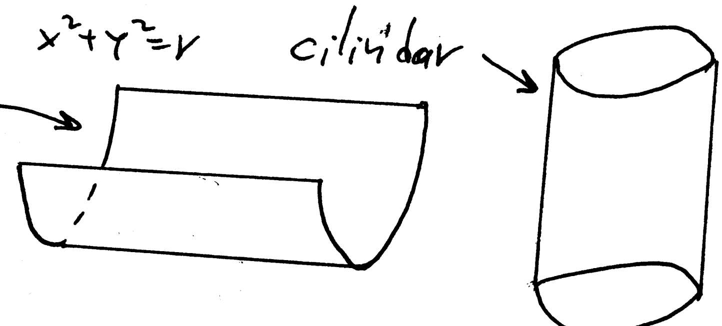
(kriva c je orjentisana pozitivno ako se posmatra sa z-ose za $z>r$).

Rj.

Izgled krive c nam u ovom zadatku neće pomoći da lakše uradimo zadatak, pa je nećemo skicirati. Samo primjetimo da je $x^2+y^2=r$ cilindar

a da je $x^2=rz$ paraboloid

i da njihov presjek proizvodi krivu c.



Prisjetimo se: Ako je

$$c: \begin{cases} x=\eta(t) \\ y=\mu(t) \\ z=\theta(t) \\ t_1 \le t \le t_2 \end{cases}$$ tada

$$\int_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \int_{t_1}^{t_2} \Big[P(\eta(t),\mu(t),\theta(t))\,\eta'(t) + Q(\eta(t),\mu(t),\theta(t))\,\mu'(t) + R(\eta(t),\mu(t),\theta(t))\,\theta'(t)\Big]\,dt$$

Da bismo parametrizirali datu krivu, prvo parametrizujmo krug $x^2+y^2=r^2$:

$$x^2+y^2=r^2: \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ 0 \le \varphi \le 2\pi \end{cases}$$

a smjene za x i y uvrstimo u paraboloid:

$x^2 = rz$

$r^2\cos^2\varphi = rz$

$z = r\cos^2\varphi$

Prema tome

$$C: \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = r\cos^2\varphi \\ 0 \le \varphi \le 2\pi \end{cases}$$

Kako je $dx = -r\sin\varphi\, d\varphi$, $dy = r\cos\varphi\, d\varphi$, $dz = r\cdot 2\cos\varphi(-\sin\varphi)\, d\varphi$ to je

$$\oint_C y\,dx + z\,dy + x\,dz = \int_0^{2\pi} \left(r\sin\varphi\cdot(-r)\sin\varphi + r\cos^2\varphi\cdot r\cos\varphi + r\cos\varphi\cdot(-2)r\sin\varphi\cos\varphi\right) d\varphi$$

$$= -\int_0^{2\pi} r^2\sin^2\varphi\, d\varphi + r^2\int_0^{2\pi}\cos^3\varphi\, d\varphi - 2r^2\int_0^{2\pi}\cos^2\varphi\sin\varphi\, d\varphi = I_1 + I_2 + I_3$$

$$I_1 = -\int_0^{2\pi} r^2\sin^2\varphi\, d\varphi = \left| \begin{array}{c} 1 = \sin^2\varphi + \cos^2\varphi \\ \cos 2\varphi = \cos^2\varphi - \sin^2\varphi \\ \hline 2\sin^2\varphi = 1 - \cos 2\varphi \\ \sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi) \end{array} \right| = -\frac{r^2}{2}\int_0^{2\pi}(1-\cos 2\varphi)\, d\varphi = \underset{\text{za vježbu}}{\ldots} = -r^2\pi$$

$$I_2 = r^2\int_0^{2\pi}\cos^3\varphi\, d\varphi = r^2\int_0^{2\pi}\cos\varphi(1-\sin^2\varphi)\, d\varphi = r^2\int_0^{2\pi}(1-\sin^2\varphi)\, d(\sin\varphi) = \underset{\text{za vježbu}}{\ldots} = 0$$

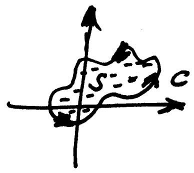
$$I_3 = -2r^2\int_0^{2\pi}\cos^2\varphi\sin\varphi\, d\varphi = \left| d(\cos\varphi) = -\sin\varphi\, d\varphi \right| = 2r^2\int_0^{2\pi}\cos^2\varphi\, d(\cos\varphi) = \underset{\text{za vježbu}}{\ldots} = 0$$

Prema tome $\oint_C y\,dx + z\,dy + x\,dz = -r^2\pi$ traženo rješenje

(#) Pomoću Greenove formule izračunati integral

$$I=\int_C (xy+x+y)\,dx+(xy+x-y)\,dy,$$ ako je c kontura kružnice $x^2+y^2=ax$ pređena u pozitivnom smislu.

Rj: Greenova formula $$\int_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$



$P(x,y)=xy+x+y$

$Q(x,y)=xy+x-y$

$\frac{\partial P}{\partial y}=x+1,\quad \frac{\partial Q}{\partial x}=y+1$

$x^2+y^2=ax$

$x^2-ax+y^2=0$

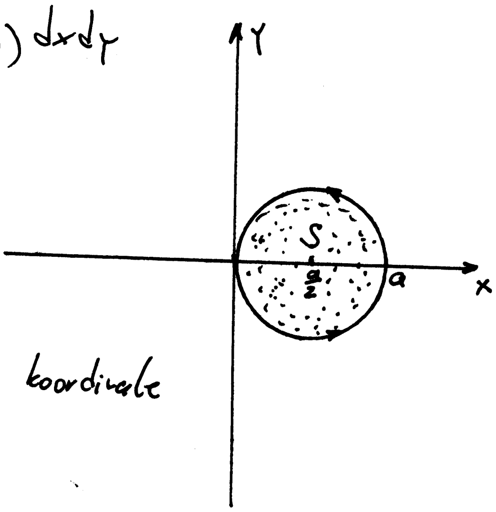
$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

krug sa centrom u $\left(\frac{a}{2},0\right)$ poluprečnika $\frac{a}{2}$

$$I=\iint_S\left(y+1-(x+1)\right)dx\,dy$$

$$I=\iint_S(y-x)\,dx\,dy$$



uvedimo polarne koordinate

$x=\frac{a}{2}+r\sin\varphi$

$y=r\cos\varphi$

$dx\,dy=r\,dr\,d\varphi$

$S \xrightarrow{\text{transformiš}} S': \begin{cases}0\le r\le\frac{a}{2}\\ 0\le\varphi\le 2\pi\end{cases}$

$$I=\iint_{S'}\left(r\cos\varphi-\frac{a}{2}+r\sin\varphi\right)r\,dr\,d\varphi=\iint_{S'}\left(r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right)dr\,d\varphi=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\frac{a}{2}}\left[r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right]dr=\int_0^{2\pi}\left[\frac{1}{3}r^3\Big|_0^{\frac{a}{2}}(\cos\varphi-\sin\varphi)-\frac{a}{2}\cdot\frac{1}{2}r^2\Big|_0^{\frac{a}{2}}\right]d\varphi$$

$$=\frac{a^3}{24}\int_0^{2\pi}(\cos\varphi-\sin\varphi)\,d\varphi-\frac{a^3}{16}\int_0^{2\pi}d\varphi=\frac{a^3}{24}\Big(\underbrace{\sin\varphi\Big|_0^{2\pi}}_{=0}+\underbrace{\cos\varphi\Big|_0^{2\pi}}_{1-1}\Big)-\frac{a^3}{16}2\pi=$$

$$=-\frac{a^3\pi}{8}$$ traženo rješenje

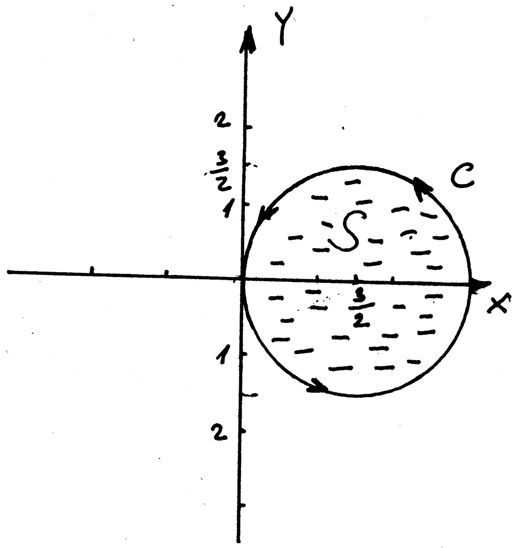
(#) Izračunati krivoliniski integral

$$I=\int_c (xy+x+y)\,dx+(xy+x-y)\,dy \quad \text{ako je } c: x^2+y^2=3x.$$

Rj.

$$x^2+y^2=3x$$
$$x^2-3x+y^2=0$$
$$x^2-2\cdot x\cdot\frac{3}{2}+\frac{9}{4}-\frac{9}{4}+y^2=0$$
$$\left(x-\frac{3}{2}\right)^2+y^2=\frac{9}{4}$$

c: Krug sa centrom u tački $\left(\frac{3}{2},0\right)$ poluprečnika $r=\frac{3}{2}$



I način: Greenova formula za ravan

$$\int_c P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

c - zatvorena kontura
S - oblast ograničena konturom

$P=xy+x+y$, $\frac{\partial P}{\partial y}=x+1$, $Q=xy+x-y$, $\frac{\partial Q}{\partial x}=y+1$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y+1-(x+1)=y-x$$

Kako je c krug, oblast ograničena krugom je unutrašnjost kruga. Da bi smo lakše opisali unutrašnjost kruga uvedimo polarne koordinate

$$x=\frac{3}{2}+r\cos\varphi$$
$$y=r\sin\varphi$$
$$dx\,dy=r\,dr\,d\varphi$$

$$D:\begin{cases}0\le r\le\frac{3}{2}\\ 0\le\varphi\le 2\pi\end{cases}$$

$$I=\int_c(xy+x+y)\,dx+(xy+x-y)\,dy=\iint_S(y-x)\,dx\,dy=\iint_D\left(r\sin\varphi-\left(\frac{3}{2}+r\cos\varphi\right)\right)\cdot r\,dr\,d\varphi$$

$$=\int_0^{3/2}\left[\int_0^{2\pi}\left(r^2\sin\varphi-\frac{3}{2}r-r^2\cos\varphi\right)d\varphi\right]dr=\int_0^{3/2}\Big(\underbrace{-r^2\cos\varphi\Big|_0^{2\pi}}_{=0}-\frac{3r}{2}\varphi\Big|_0^{2\pi}\underbrace{-r\sin\varphi\Big|_0^{2\pi}}_{=0}\Big)dr$$

$$=\int_0^{3/2}-3\pi r\,dr=-3\pi\frac{r^2}{2}\Big|_0^{3/2}=-\frac{3}{2}\pi\cdot\frac{9}{4}=-\frac{27}{8}\pi$$

II način: Klasičan način

C kriva u ravni opisana jednačinom $y=\eta(x)$, $a \le x \le b$

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \left[P(x,\eta(x)) + Q(x,\eta(x))\cdot \eta'(x)\right]dx$$

Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1 \le t \le t_2$ tada

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_{t_1}^{t_2} \left[P(\mu(t),\eta(t))\mu'(t) + Q(\mu(t),\eta(t))\eta'(t)\right]dt$$

U našem slučaju c je kružnica. Parametarizirajmo kružnicu

$x = \frac{3}{2} + r\cos\varphi$

$y = r\sin\varphi$

U našem slučaju $r=\frac{3}{2}$ a umjesto promjenjive φ stavimo promjenjivu t

$x = \frac{3}{2} + \frac{3}{2}\cos t$

$y = \frac{3}{2}\sin t$ gdje $0 \le t \le 2\pi$

$\frac{\partial x}{\partial t} = -\frac{3}{2}\sin t$

$\frac{\partial y}{\partial t} = \frac{3}{2}\cos t$

$$I = \int_C (xy+x+y)\,dx + (xy+x-y)\,dy = \int_0^{2\pi} \Big[\Big(\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)\big(\tfrac{3}{2}\sin t\big) + \big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big) + \big(\tfrac{3}{2}\sin t\big)\Big)\big(-\tfrac{3}{2}\sin t\big) + \Big(\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)\big(\tfrac{3}{2}\sin t\big) + \big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big) - \big(\tfrac{3}{2}\sin t\big)\Big)\tfrac{3}{2}\cos t\Big]dt = \ldots$$

na klasičan način ovo je komplikovano ali se može izračunati

$$I = -\frac{27}{8}\pi$$

(#) Izračunati pomoću Greenove formule krivolinijski integral

$$I=\oint_C \left(x^2y+\frac{1}{3}y^3+ye^{xy}\right)dx+\left(x+xe^{xy}\right)dy$$

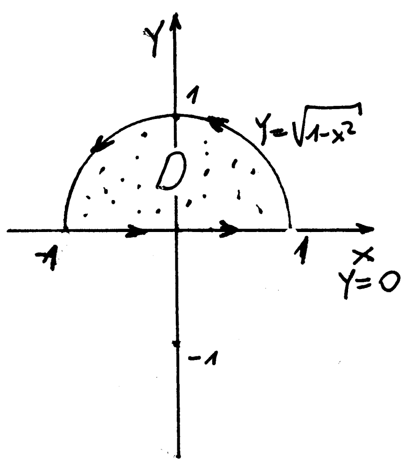
ako je c pozitivno orjentisana kontura određena linijama $y=\sqrt{1-x^2}$, $y=0$.

Rj. Greenova formula glasi $\oint_C P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$

gdje je c-zatvorena kontura, S-oblast ograničena konturom

$y=\sqrt{1-x^2}$ $\qquad y^2=1-x^2$

$x^2+y^2=1$



$$P=x^2y+\frac{1}{3}y^3+ye^{xy}$$

$$\frac{\partial P}{\partial y}=x^2+y^2+e^{xy}+ye^{xy}\cdot x$$

$$Q=x+xe^{xy}$$

$$\frac{\partial Q}{\partial x}=1+e^{xy}+xe^{xy}\cdot y$$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=1-x^2-y^2$$

$$\oint_C\left(x^2y+\frac{1}{3}y^3+ye^{xy}\right)dx+\left(x+xe^{xy}\right)dy=\left|\begin{matrix}\text{primjena}\\ \text{Greenove}\\ \text{formule}\end{matrix}\right|=\iint_D\left(1-x^2-y^2\right)dx\,dy$$

$$=\left|\begin{matrix}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi\end{matrix}\quad D\xrightarrow{\text{transformise}}D':\begin{cases}0\le r\le 1\\ 0\le\varphi\le\pi\end{cases}\right|=\iint_{D'}\left(1-r^2\right)r\,dr\,d\varphi$$

$x^2+y^2=r^2$

$$=\int_0^1\left(r-r^3\right)dr\int_0^\pi d\varphi=\varphi\Big|_0^\pi\int_0^1\left(r-r^3\right)dr=\pi\left(\frac{1}{2}r^2\Big|_0^1-\frac{1}{4}r^4\Big|_0^1\right)=\frac{\pi}{4}$$

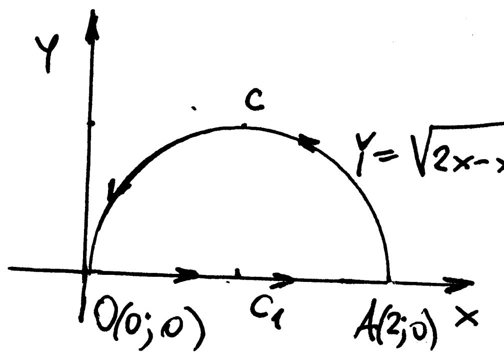
traženo rješenje

Izračunati

$$I=\int_C (e^{x+y}\sin 2y+x+y)\,dx+(e^{x+y}(2\cos 2y+\sin 2y)+2x)\,dy$$

gdje je C kriva $y=\sqrt{2x-x^2}$, integracija se vrši od tačke $A(2;0)$ do tačke $O(0;0)$.

Rj. Skicirajmo krivu $y=\sqrt{2x-x^2}$.



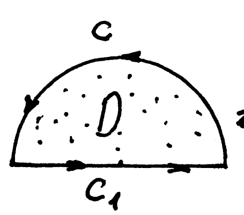
$y^2=2x-x^2$

$x^2-2\cdot x\cdot 1+1+y^2=1$

$(x-1)^2+y^2=1$

krug sa centrom u $C(1;0)$ polupr. $r=1$

Označimo sa c_1 krivu (pravu) od tačke $O(0;0)$ do $A(2;0)$, i označimo sa $P(x,y)=e^{x+y}\sin 2y+x+y$, $Q(x,y)=e^{x+y}(2\cos 2y+\sin 2y)+2x$. Neka je J integral u kome se integracija vrši prvo po krivoj c od tačke $A(2;0)$ do tačke $O(0;0)$ pa onda po krivoj c_1 od tačke $O(0;0)$ do tačke $A(2;0)$



$$J=\int_{cc_1} P(x,y)\,dx+Q(x,y)\,dy=\underbrace{\int_c P\,dx+Q\,dy}_{=I}+\int_{c_1} P\,dx+Q\,dy$$

$$\Rightarrow\quad I=J-\overbrace{\int_{c_1} P\,dx+Q\,dy}^{=J_1}$$

~~Zašto~~ ovo?

Primjetimo da integral J možemo izračunati pomoću formule Greena

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\left.\begin{aligned}\frac{\partial Q}{\partial x} &= e^{x+y}(2\cos 2y + \sin 2y) + 2\\ \frac{\partial P}{\partial y} &= e^{x+y}\sin 2y + e^{x+y}\, 2\cos 2y + 1\end{aligned}\right\} \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

$$J = \iint_D dx\,dy = \left| \begin{array}{l}\text{kako je } D \text{ polukrug}\\ \text{a znamo da se}\\ \text{površina kruga računa}\\ \text{po formuli } r^2\pi\end{array} \right| = \frac{1}{2}\pi$$

$$J_1 = \left| C_1 : \begin{cases} y = 0 \\ 0 \le x \le 2 \end{cases} \right| = \int_0^2 x\,dx = \frac{1}{2}x^2 \Big|_0^2 = 2$$

Prema tome

$$I = J - J_1 = \frac{\pi}{2} - 2$$ traženo rješenje

(#) Izračunati krivolinijski integral

$$I = \int_{\widehat{AO}} (e^x \sin y - my)\,dx + (e^x \cos y - m)\,dy$$

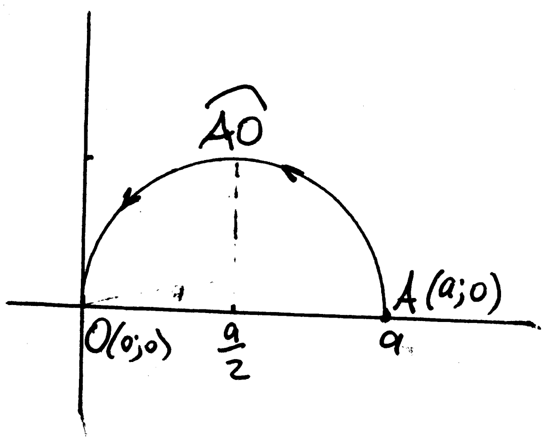
gdje je $\widehat{AO}$ gornji polukrug $x^2+y^2=ax$, $y \geq 0$ $(a>0)$ orjentisan od tačke $A(a;0)$ do tačke $O(0;0)$.

Rj. $x^2+y^2=ax$

$$x^2 - 2\cdot x \cdot \frac{a}{2} + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + y^2 = 0$$

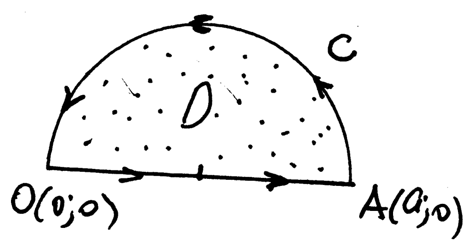
$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

krug sa centrom u $C\left(\frac{a}{2}, 0\right)$ poluprečnika $r = \frac{a}{2}$



Primjetimo da kriva $\widehat{AO}$ nije zatvorena, pa ne možemo primjeniti formulu Greena. Međutim, ako zatvorimo polukrug $\widehat{AO}$ sa duži $\overline{OA}$ dobićemo integral

$$J = \int_{C} (e^x \sin y - my)\,dx + (e^x \cos y - m)\,dy$$



na koji možemo upotrebiti formulu Greena. Sada primjetimo

$$J = I + \int_{\overline{OA}} (e^x \sin y - my)\,dx + (e^x \cos y - m)\,dy$$

tj.

$$I = J - \underbrace{\int_{\overline{OA}} (e^x \sin y - my)\,dx + (e^x \cos y - m)\,dy}_{=J_1}$$

Greenova formula

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Izračunajmo

$$J = \int_C \underbrace{(e^x \sin y - my)}_{P}\,dx + \underbrace{(e^x \cos y - m)}_{Q}\,dy$$

primjenom Greenove formule

$$\frac{\partial P}{\partial y} = e^x \cos y - m \qquad \frac{\partial Q}{\partial x} = e^x \cos y$$

$$J = \iint_D m\,dx\,dy = m \underbrace{\iint_D dx\,dy}_{\substack{\text{površina polukruga} \\ P = r^2\pi}} = m \cdot \frac{1}{2}\left(\frac{a}{2}\right)^2 \pi = \frac{1}{8} a^2 m \pi$$

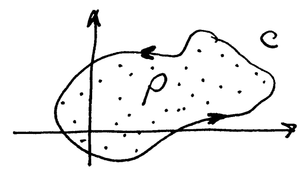
$$J_1 = \left| \overline{OA} : \begin{cases} y = 0 \\ 0 \le x \le a \end{cases} \right| = \int_0^a (e^x \cdot 0 - m \cdot 0)\,dx + (e^x \cdot 1 - m) \cdot 0 = 0$$

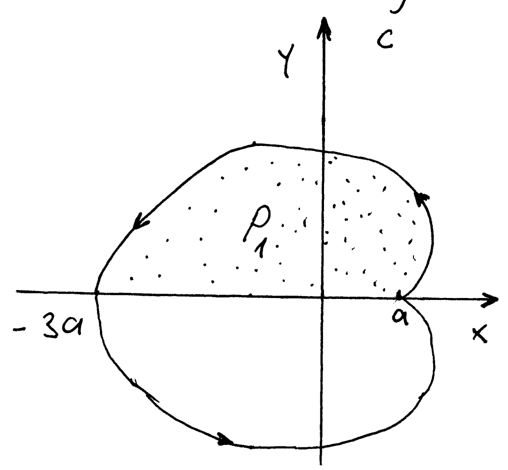
Prema tome $I = J - J_1 = \frac{1}{8} a^2 m \pi$ traženo rješenje

(#) Uz pomoć krivolinijskog integrala druge vrste, izračunati površinu, ograničenu kardioidom $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$.

Rj. Prisjetimo se, površina figure ograničene krivom c se računa po formuli

$$P = \frac{1}{2}\oint_c x\,dy - y\,dx$$





kardioida

$x = 2a\cos t - a\cos 2t$

$y = 2a\sin t - a\sin 2t$

$t=0: \; x=a, \; y=0$

$t=\pi: \; x=-3a, \; y=0$

Primjetimo da je kardioida kriva linija koja je simetrična u odnosu na x-osu, pa da bi izračunali površinu ograničenu kardioidom dovoljno je izračunati površinu iznad x-ose

Da bi smo opisali kardioidu parametar t uzima vrijednosti od 0 do 2π.

Prisjetimo se, ako je kriva c data u parametarskom obliku $x = \mu(t)$, $y = \eta(t)$, $t_1 \le t \le t_2$ tada se krivolinijski integral računa po formuli

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \int_{t_1}^{t_2} \left(P(\mu(t), \eta(t))\,\mu'(t) + Q(\mu(t), \eta(t))\,\eta'(t) \right) dt$$

$$P = \frac{1}{2}\oint_c x\,dy - y\,dx = \left| \begin{matrix} x = 2\cos t - \cos 2t \\ dx = (-2\sin t + 2\sin 2t)\,dt \\ y = 2\sin t - \sin 2t \\ dy = (2\cos t - 2\cos 2t)\,dt \end{matrix} \right| = \frac{1}{2}\int_0^{2\pi} (2\cos t - \cos 2t)\cdot(2\cos t - 2\cos 2t$$

$$= \frac{1}{2}\int_0^{2\pi} \left[(2\cos t - \cos 2t)(2\cos t - 2\cos 2t) - (2\sin t - \sin 2t)(-2\sin t + 2\sin 2t) \right] dt = 2P_1$$

$$= \int_0^{\pi} \left(\underline{4\cos^2 t} - 6\cos t\cos 2t + \underline{\underline{2\cos^2 2t}} + \underline{4\sin^2 t} - 6\sin t\sin 2t + \underline{\underline{2\sin^2 2t}} \right) dt =$$

$$= \int_0^{\pi} (6 - 6\cos t\cos 2t - 6\sin t\sin 2t)\,dt = 6\int_0^{\pi} (1 - \cos(t - 2t))\,dt = \ldots = 6\pi$$

Izračunati pomoću krivolinijskog integrala II vrste površinu ravne figure ograničene konturom

$$c: \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \\ 0 \leq t \leq 2\pi \end{cases}.$$

Rj. Površina figure ograničenu zatvorenom linijom c računamo po formuli

$$P = \frac{1}{2}\int_c x\,dy - y\,dx.$$

$$x = a(t - \sin t) \qquad y = a(1 - \cos t)$$

$$dx = a(1 - \cos t) \qquad dy = a \sin t$$

$$x\,dy - y\,dx = a(t - \sin t)\cdot a\sin t - a(1-\cos t)\cdot a(1-\cos t)$$

$$= a^2 t\sin t - a^2\sin^2 t - a^2(1-\cos t)^2$$

$$= a^2(t\sin t - \sin^2 t - 1 + 2\cos t - \cos^2 t)$$

$$= a^2(t\sin t + 2\cos t - 2)$$

$$P = \frac{1}{2}\int_c x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(a^2(t\sin t + 2\cos t - 2))\,dt =$$

$$= \frac{a^2}{2}\left(\int_0^{2\pi} t\sin t\,dt + 2\int_0^{2\pi}\cos t\,dt - 2\int_0^{2\pi} dt\right) = \ldots = \frac{a^2}{2}(-2\pi + 0 - 4\pi) = 3a^2\pi$$

(#) Izračunati krivolinijski integral $\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$ duž puta

koji ne prolazi kroz koordinatni početak.

Rj: Ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ tada vrijednost integrala $\int_c P\,dx + Q\,dy$

ne zavisi od vrste izbora puta integracije.

$$I = \int_{(1,0)}^{(6,8)} \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy \Rightarrow \left. \begin{array}{l} P(x,y) = \frac{x}{\sqrt{x^2+y^2}} \\ Q(x,y) = \frac{y}{\sqrt{x^2+y^2}} \end{array} \right\} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -\frac{xy}{\sqrt{(x^2+y^2)^3}}$$

Prema tome vrijednost integrala ne zavisi od izbora krive kojom ćemo spojiti tačke (1,0) i (6,8).

I način: Odrediti ćemo primitivnu f-ju u.

$u = u(x,y)$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$du = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$

$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \quad \ldots(1)$

$\partial u = \frac{x}{\sqrt{x^2+y^2}}\partial x$

$$u = \int \frac{x}{\sqrt{x^2+y^2}}dx + \varphi(y) = \left| \begin{array}{c} x^2+y^2 = t^2 \\ 2x\,dx = 2t\,dt \\ x\,dx = t\,dt \end{array} \right| = \int \frac{t}{\sqrt{t^2}}dt + \varphi(y)$$

$$= t + \varphi(y) = \sqrt{x^2+y^2} + \varphi(y)$$

$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} + \varphi'(y) \quad \ldots(2)$

(1) i (2) $\Rightarrow \varphi'(y) = 0 \Rightarrow u = \sqrt{x^2+y^2}$

$$\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \int_{(1,0)}^{(6,8)} du = u\Big|_{(1,0)}^{(6,8)} = \sqrt{x^2+y^2}\Big|_{(1,0)}^{(6,8)} = \sqrt{36+64} - \sqrt{1+0} = 9$$

II način: Spojimo tačke (1,0) i (6,8) nekom krivom koja ne prolazi kroz koordinatni početak i izračunamo integral na klasičan način.

Izračunati krivoliniski integral $\int_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2}$ duž putanje koja ne siječe osu Oy.

Rj:

Vrijednost integrala $I = \int_c P(x,y)\,dx + Q(x,y)\,dy$ ne zavisi od vrste konture c ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

U našem slučaju $I = \int_{(2,1)}^{(1,2)} \frac{y}{x^2}\,dx - \frac{1}{x}\,dy$ $\quad P(x,y) = \frac{y}{x^2}$, $Q(x,y) = -\frac{1}{x}$

$$\frac{\partial P}{\partial y} = \frac{1}{x^2}, \quad \frac{\partial Q}{\partial x} = \frac{1}{x^2}$$

Prema tome vrijednost integrala ne zavisi od vrste krive linije c koja spaja tačke (2,1) i (1,2).

I način: Odredićemo primitivnu f-ju

$P(x,y)\,dx + Q(x,y)\,dy = 0$ $\quad$ $u = u(x,y)$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\quad$ ovo je egzaktna dif. jednačina $\quad$ $du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$

$$du = \frac{y}{x^2}\,dx - \frac{1}{x}\,dy$$

$\partial u = \frac{y}{x^2}\,\partial x$ $\quad$ $\frac{\partial u}{\partial x} = \frac{y}{x^2}, \quad \frac{\partial u}{\partial y} = -\frac{1}{x} \quad \ldots(1)$

$$u = \int \frac{y}{x^2}\,dx + \varphi(y) = y\,\frac{x^{-1}}{-1} + \varphi(y) = -\frac{y}{x} + \varphi(y)$$

$\frac{\partial u}{\partial y} = -\frac{1}{x} + \varphi'(y) \quad \ldots(2)$ $\quad$ (1) i (2) $\Rightarrow \varphi'(y) = 0$, $\varphi(y) = c$

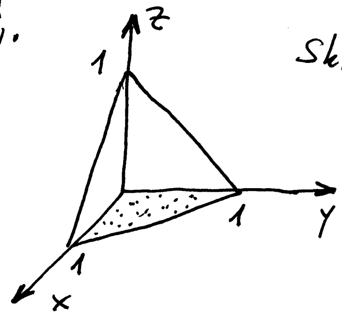
$$u = -\frac{y}{x} + c$$

$$\int_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2} = \int_{(2,1)}^{(1,2)} du = -\frac{y}{x}\Big|_{(2,1)}^{(1,2)} = -\frac{2}{1} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 2 = -\frac{3}{2}$$

II način: Spojimo tačke (2,1) i (1,2) nekom krivom (ili pravom) ili izlomljenom pravom linijom i izračunamo integral na klasičan način.

(#) Izračunati površinski integral $I = \int_S xyz\,dS$, ako je S dio ravni $x+y+z=1$ u I oktantu.

Rj.



Skiciramo $x+y+z=1$.

Ako je D projekcija površi S, koja je opisana $z=z(x,y)$, na xOy ravan tada

$$\int_S f(x,y,z)\,dS = \int_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

U našem slučaju S je opisa formulom $z=1-x-y$.

Ako S projiciramo na xOy ravan dobićemo $D: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$

$$\int_S xyz\,dS = \iint_D xy(1-x-y)\,dx\,dy = \int_0^1 dx \int_0^{1-x} (xy - x^2y - xy^2)\,dy =$$

$$\frac{\partial z}{\partial x} = -1 \qquad \frac{\partial z}{\partial y} = -1 \qquad \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3}$$

$$= \int_0^1 \left(\frac{1}{2}xy^2\Big|_0^{1-x} - \frac{1}{2}x^2y^2\Big|_0^{1-x} - \frac{1}{3}xy^3\Big|_0^{1-x}\right)dx =$$

$$= \int_0^1 (1-x)^2\left(\frac{1}{2}x - \frac{1}{2}x^2 - \underbrace{\frac{1}{3}x(1-x)}_{-\frac{1}{3}x+\frac{1}{3}x^2}\right)dx = \int_0^1 (1-x)^2\left(-\frac{1}{6}x^2 + \frac{1}{6}x\right)dx$$

$$= -\frac{1}{6}\int_0^1 (1-x)^2 \cdot \underbrace{(x^2-x)}_{x(x-1)}\,dx = \frac{1}{6}\int_0^1 (1-x)^3 x\,dx = \left| \begin{array}{ll} 1-x=t & x\Big|_0^1 \to \\ -dx=dt & y\Big|_1^0 \\ dx=-dt & \\ x=1-t & \end{array} \right|$$

$$= \frac{1}{6}\int_0^1 t^3(1-t)\,dt = \frac{1}{6}\int_0^1 (t^3-t^4)\,dt = \frac{1}{6}\cdot\frac{1}{4}t^4\Big|_0^1 - \frac{1}{6}\cdot\frac{1}{5}t^5\Big|_0^1$$

$$= \frac{1}{24} - \frac{1}{30} = \frac{30-24}{24\cdot 30} = \frac{6}{24\cdot 30} = \frac{1}{24\cdot 5} = \frac{1}{120}$$ traženi rezultat

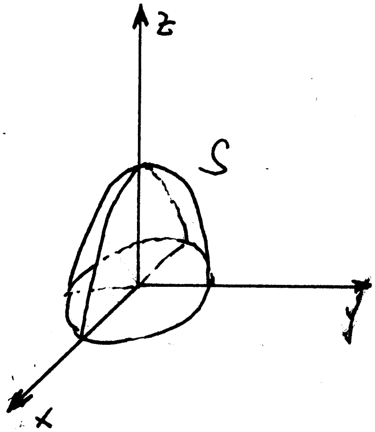
(#) Izračunati površinski integral $\iint_S 3z\,dS$ gdje je S površina paraboloida $z = 2-(x^2+y^2)$ iznad xy-ravni.

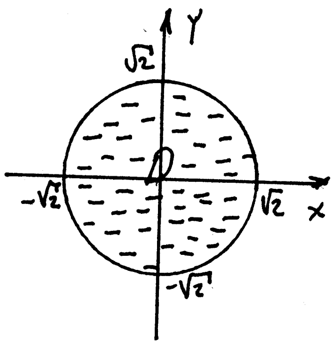
R: Neka je D projekcija površi S na xO_y ravan. Tada

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

Pronađimo projekciju paraboloida $z=2-(x^2+y^2)$ na xO_y ravan.

$z=0 \Rightarrow x^2+y^2=2$ krug sa centrom u tački $(0,0)$ poluprečnika $\sqrt{2}$





$$\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial y} = -2y$$

$$I = \iint_S 3z\,dS = 3\iint_D \left[2-(x^2+y^2)\right]\sqrt{1+4x^2+4y^2}\,dx\,dy$$

Da bi smo riješili ovaj dvostruki integral potrebno je uvesti smjenu promjenjivih.

Uvedimo polarne koordinate $x = r\cos\varphi$, $y = r\sin\varphi$

$$D': \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le \sqrt{2} \\ dx\,dy = r\,dr\,d\varphi \end{cases}$$

ove granice čitamo sa slike

$$x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$$

$$1+4x^2+4y^2 = 1+4(x^2+y^2) = 1+4r^2$$

$$I = 3\iint_{D'}(2-r^2)\sqrt{1+4r^2}\cdot r\cdot dr\,d\varphi = 3\iint_{D'} 2r\sqrt{1+4r^2}\,dr\,d\varphi - 3\iint_{D'} r^3\sqrt{1+4r^2}\,dr\,d\varphi$$

$$6\iint_{D'} r\sqrt{1+4r^2}\,dr\,d\varphi = 6\int_0^{2\pi}\left[\int_0^{\sqrt{2}} r\sqrt{1+4r^2}\,dr\right]d\varphi = \left|\begin{array}{ll} 1+4r^2=t^2 & r=0 \Rightarrow t=1 \\ 8r\,dr=2t\,dt & r=\sqrt{2} \Rightarrow t=3 \\ r\,dr=\frac{1}{4}t\,dt & \end{array}\right| =$$

$$= 6\int_0^{2\pi}\left[\int_1^3 \frac{1}{4}t^2\,dt\right]d\varphi = 6\cdot\frac{1}{4}\,\varphi\Big|_0^{2\pi}\cdot\frac{t^3}{3}\Big|_1^3 = \frac{3}{2}\cdot\frac{1}{3}\cdot 2\pi\cdot 26 = 26\pi$$

$$3\iint_{D'} r^3\sqrt{1+4r^2}\,dr\,d\varphi = 3\int_0^{2\pi}\left[\int_0^{\sqrt{2}} \underbrace{r^3}_{r^2\cdot r}\sqrt{1+4r^2}\,dr\right]d\varphi = \left|\begin{array}{ll} 1+4r^2=t^2 & r\,dr=\frac{1}{4}t\,dt \\ 4r^2=t^2-1 & r=0 \Rightarrow t=1 \\ r^2=\frac{1}{4}(t^2-1) & r=\sqrt{2} \Rightarrow t=3 \\ 8r\,dr=2t\,dt & \end{array}\right| =$$

$$= 3\int_0^{2\pi}\left[\int_1^3 \frac{1}{16}(t^2-1)\,t\cdot t\,dt\right]d\varphi = \frac{3}{16}\int_0^{2\pi}\left[\int_1^3 (t^4-t^2)\,dt\right]d\varphi = \frac{3}{16}\cdot\varphi\Big|_0^{2\pi}\cdot\left(\frac{1}{5}t^5\Big|_1^3 - \frac{1}{3}t^3\Big|_1^3\right)$$

$$= \frac{3}{8}\pi\cdot\left(\frac{242}{5} - \frac{26}{3}\right) = \frac{1}{8}\pi\left(\frac{726}{5} - 26\right) = \frac{1}{8}\pi\,\frac{726-130}{5} = \frac{596\pi}{40} = \frac{149\pi}{10}$$

$$\iint_S 3z\,dS = 26\pi - \frac{149\pi}{10} = \frac{260-149}{10}\pi = \frac{111\pi}{10}$$

traženo rješenje

(#) Izračunati površinski integral $\iint\limits_{(S)} \sqrt{-x^2+4}\, dS$, gdje je (S) omotač površi $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0 \le z \le 3$.

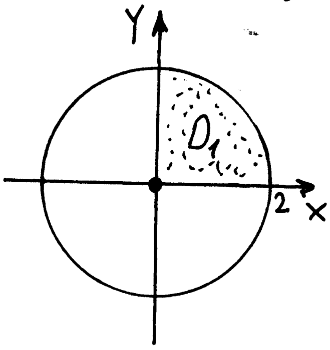
Rj. Skicirajmo površ $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0 \le z \le 3$

u xOy ravni

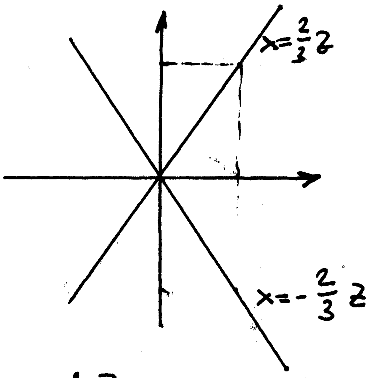
$$\frac{x^2}{4}+\frac{y^2}{4}=0$$

za $z=0$, $x^2+y^2=0$

tačka $(0,0)$



u xOz ravni



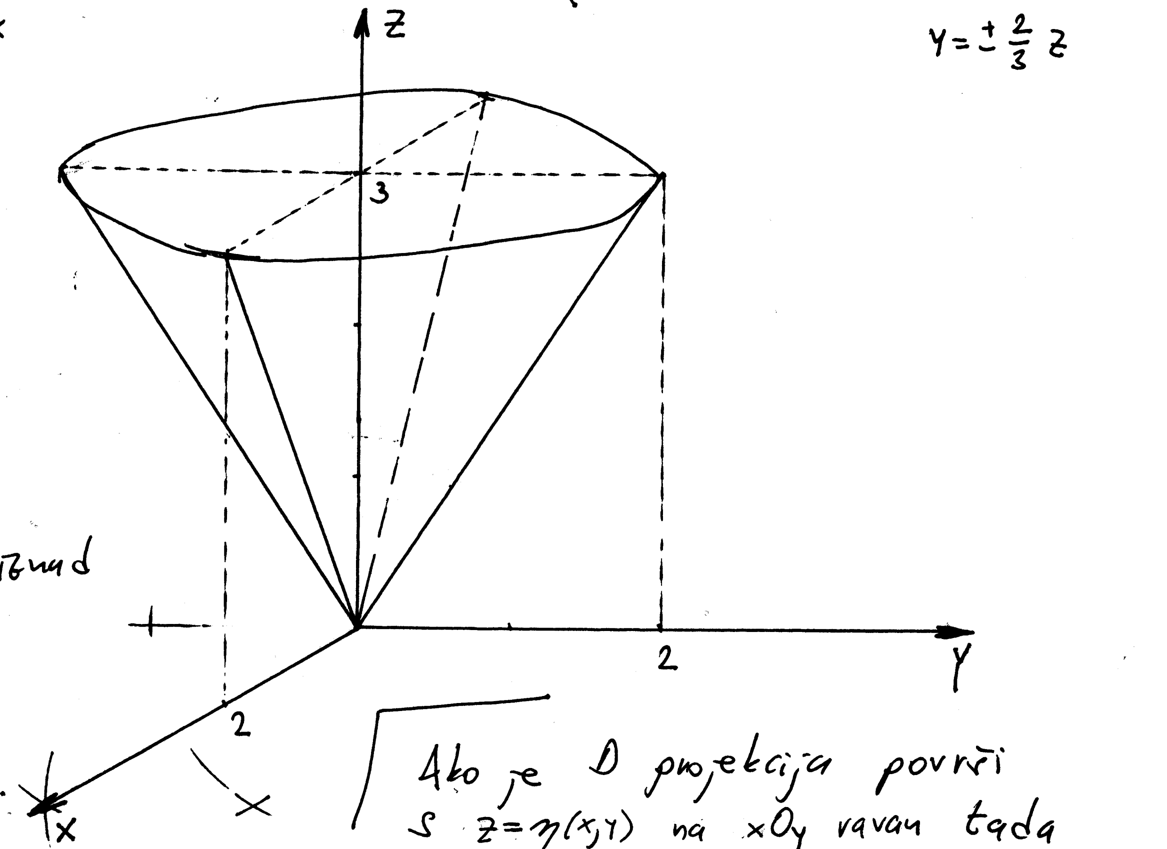
$$\frac{x^2}{4}=\frac{z^2}{9}$$

$$x^2=\frac{4}{9}z^2$$

$$x=\pm\frac{2}{3}z$$

yOz ravan

$$y=\pm\frac{2}{3}z$$



za $z=3$ $x^2+y^2=4$

$$\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$$

$$z^2=\frac{9}{4}(x^2+y^2)$$

Kako je data površ iznad xOy ravni

$$z=\frac{3}{2}\sqrt{x^2+y^2}$$

$$z'_x=\frac{3}{2}\cdot\frac{2x}{2\sqrt{x^2+y^2}}=\frac{3x}{2\sqrt{x^2+y^2}} \qquad z'_y=\frac{3y}{2\sqrt{x^2+y^2}}$$

Ako je D projekcija površi S $z=\eta(x,y)$ na xOy ravan tada

$$\iint\limits_S f(x,y,z)\,dS=\iint\limits_D f(x,y,\eta(x,y))\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{9x^2}{4(x^2+y^2)}+\frac{9y^2}{4(x^2+y^2)}=\frac{13x^2+13y^2}{4(x^2+y^2)}=\frac{13}{4}$$

Primjetimo da je data površ (S) simetrična u odnosu na xOz ravan i yOz ravan pa možemo pisati

$$\iint_{(S)} \sqrt{-x^2+4}\, dS = \frac{\sqrt{13}}{2} \iint_{D} \sqrt{-x^2+4}\, dx\,dy = 4 \cdot \frac{\sqrt{13}}{2} \iint_{D_1} \sqrt{4-x^2}\, dx\,dy$$

gdje je $D_1 : \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \end{cases}$

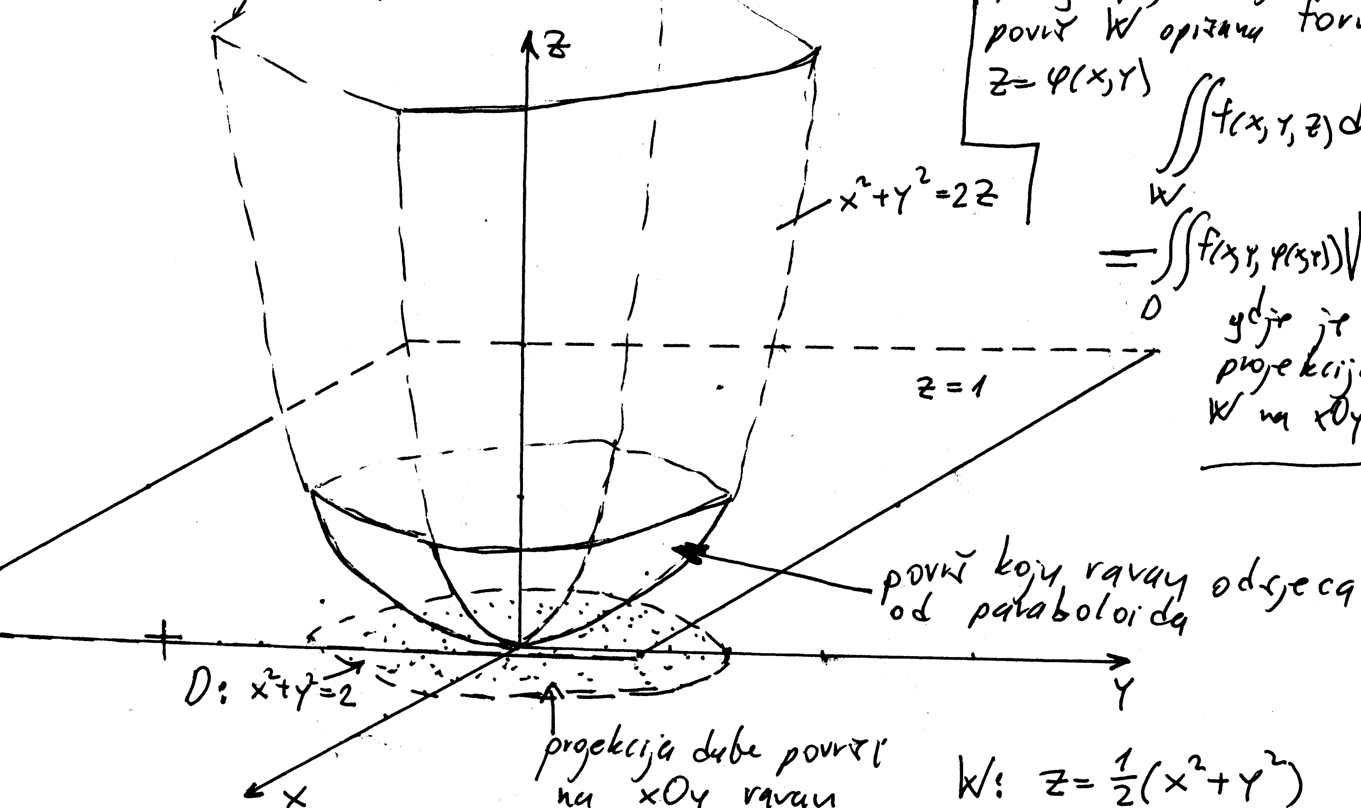
$$\iint_{(S)} \sqrt{-x^2+4}\, dS = 2\sqrt{13} \int_0^2 \sqrt{4-x^2}\, dx \int_0^{\sqrt{4-x^2}} dy = 2\sqrt{13} \int_0^2 (4-x^2)\, dx =$$

$$= 2\sqrt{13} \left(4x \Big|_0^2 - \frac{1}{3} x^3 \Big|_0^2 \right) = 2\sqrt{13} \left(8 - \frac{8}{3} \right) = 2\sqrt{13} \cdot \frac{\overset{16}{24-8}}{3}$$

$$= \frac{32}{3} \sqrt{13}$$ traženo rješenje

(#) Izračunati površinski integral prvog tipa $\iint_W (x^2+y^2)\,dS$, gdje je W – površina dijela paraboloida $x^2+y^2=2z$ koju odsjeca ravan $z=1$ (dio paraboloida ispod date ravni).

Rj. Skicirajmo paraboloid i ravan



Prisjetimo se kako se računa površinski integral prvog tipa, ako je data površ W opisana formulom $z=\varphi(x,y)$

$$\iint_W f(x,y,z)\,dS = \iint_D f(x,y,\varphi(x,y))\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}\,dx\,dy$$

gdje je D projekcija površi W na xOy ravan

$W:\ z=\frac{1}{2}(x^2+y^2)$

$\frac{\partial z}{\partial x}=x,\quad \frac{\partial z}{\partial y}=y$

$$\iint_W (x^2+y^2)\,dS = \iint_D (x^2+y^2)\sqrt{1+x^2+y^2}\,dx\,dy = \left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi \quad D \xrightarrow{\text{transf.}} D': \begin{cases}0\le r\le\sqrt{2}\\ 0\le\varphi\le 2\pi\end{cases}\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi \qquad x^2+y^2=r^2\end{array}\right.$$

$$=\iint_{D'} r^2\sqrt{1+r^2}\,r\,dr\,d\varphi = \int_0^{2\pi}d\varphi\int_0^{\sqrt{2}} r^2\sqrt{1+r^2}\,r\,dr = \left|\begin{array}{ll}1+r^2=t^2 & r^2=t^2-1\\ 2r\,dr=2t\,dt & \\ r\,dr=t\,dt & r\Big|_0^{\sqrt{2}} \Rightarrow t\Big|_1^{\sqrt{3}}\end{array}\right|$$

$$=\int_0^{2\pi}d\varphi\int_1^{\sqrt{3}}(t^2-1)\sqrt{t^2}\,t\,dt = \int_0^{2\pi}d\varphi\int_1^{\sqrt{3}}(t^4-t^2)\,dt = \varphi\Big|_0^{2\pi}\left(\frac{1}{5}t^5\Big|_1^{\sqrt{3}}-\frac{1}{3}t^3\Big|_1^{\sqrt{3}}\right)=$$

$$=2\pi\left(\frac{9\sqrt{3}-1}{5}-\frac{3\sqrt{3}-1}{3}\right)=2\pi\,\frac{27\sqrt{3}-3-15\sqrt{3}+5}{15}=2\pi\,\frac{12\sqrt{3}+2}{15}=\frac{(24\sqrt{3}+4)\pi}{15}$$ traženo rješenje

(#) Izračunati površinski integral

$$I = \iint_S \frac{dS}{(1+z)^2}$$

ako je S sfera $x^2+y^2+z^2=1$.

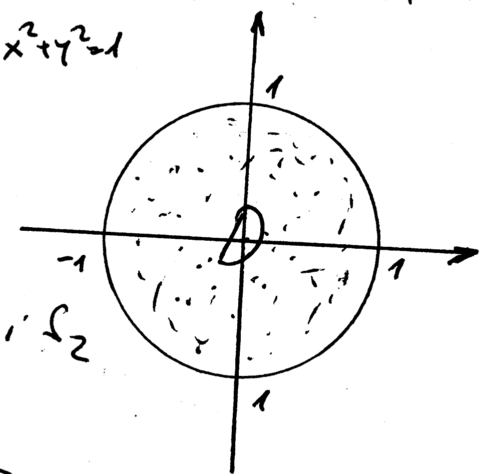
Rj. Zadatak se može uraditi na više načina

I način

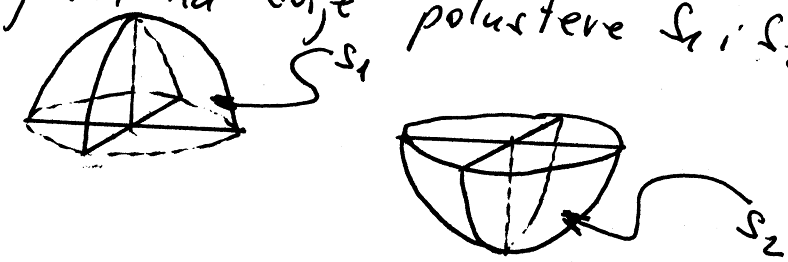
$z^2 = 1-x^2-y^2$

$z = \pm\sqrt{1-x^2-y^2}$

Presjek sfere $x^2+y^2+z^2=1$ sa xOy-ravni je krug $x^2+y^2=1$



U ovom slučaju sferu S ćemo podijeliti na dvije polusfere S_1 i S_2



$$I = \iint_S \frac{dS}{(1+z)^2} = \iint_{S_1} \frac{dS_1}{(1+z)^2} + \iint_{S_2} \frac{dS_2}{(1+z)^2}$$

gdje je $S_1: z = \sqrt{1-x^2-y^2}$

a $S_2 = -\sqrt{1-x^2-y^2}$

Znamo da je $dS_1 = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1-x^2-y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2}}, \quad dS_1 = \sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\,dx\,dy$$

$$dS_1 = \sqrt{\frac{1}{1-x^2-y^2}} = \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$$

Sad imamo

$$\iint_{S_1} \frac{dS_1}{(1+z)^2} = \iint_D \frac{1}{(1+\sqrt{1-x^2-y^2})^2} \cdot \frac{1}{\sqrt{1-x^2-y^2}}\, dx\,dy =$$

$$= \left| \begin{array}{l} \text{uvodimo polarne koordinate} \\ x=\rho\cos\varphi \\ y=\rho\sin\varphi \\ dx\,dy=\rho\, d\rho\, d\varphi \end{array} \quad D \xrightarrow{\text{transf.}} D_1 : \begin{cases} 0\le \rho \le 1 \\ 0\le \varphi \le 2\pi \end{cases} \quad x^2+y^2=\ldots=\rho^2 \right| = \iint_{D_1} \frac{\rho\, d\rho\, d\varphi}{(1+\sqrt{1-\rho^2})^2\sqrt{1-\rho^2}}$$

$$= \int_0^1 \frac{\rho\, d\rho}{(1+\sqrt{1-\rho^2})^2\sqrt{1-\rho^2}} \int_0^{2\pi} d\varphi = \left| \begin{array}{l} 1-\rho^2=t^2 \\ -2\rho\, d\rho = 2t\,dt \\ \rho\, d\rho = -t\,dt \end{array} \right| = 2\pi \int_1^0 \frac{-t\,dt}{(1+t)^2\cdot t}\, dt = \ldots = \pi$$

Slično

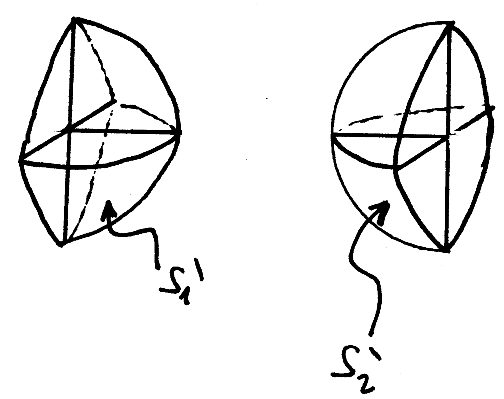
$$\iint_{S_2} \frac{dS_2}{(1+z)^2} = \iint_D \frac{1}{(1-\sqrt{1-x^2-y^2})} \cdot \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \left| \begin{array}{l} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{array} \right| = \iint_{D_1} \frac{\rho\, d\rho\, d\varphi}{(1-\sqrt{1-\rho^2})^2\sqrt{1-\rho^2}}$$

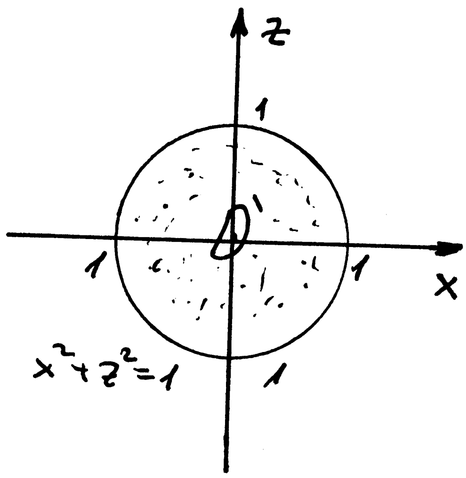
$$= 2\pi \int_0^1 \frac{\rho\, d\rho}{(1-\sqrt{1-\rho^2})^2\sqrt{1-\rho^2}} = \left| \begin{array}{l} 1-\rho^2=t^2 \\ -2\rho\, d\rho = 2t\,dt \\ \rho\, d\rho = -t\,dt \end{array} \right| = 2\pi \int_0^1 \frac{t\,dt}{(1-t)^2 t} = \ldots = \infty$$

II način

$$y^2 = 1-x^2-z^2$$

$$y = \pm\sqrt{1-x^2-z^2}$$





$$I = \iint_S \frac{dS}{(1+z)^2} = \iint_{S_1'} \frac{dS_1'}{(1+z)^2} + \iint_{S_2'} \frac{dS_2'}{(1+z)^2} \quad \text{gdje je} \quad S_1' : y=\sqrt{1-x^2-z^2}, \quad S_2' : y=-\sqrt{1-x^2-z^2}$$

$$y=\sqrt{1-x^2-z^2}, \quad \frac{\partial y}{\partial x} = \frac{-x}{\sqrt{1-x^2-z^2}}, \quad \frac{\partial y}{\partial z} = \frac{-z}{\sqrt{1-x^2-z^2}}, \quad 1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2 = \frac{1}{1-x^2-z^2}$$

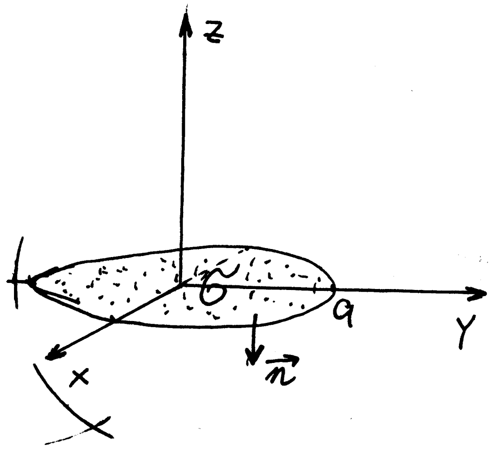
$$\iint_{S_1'} \frac{dS_1'}{(1+z)^2} = \iint_{D'} \frac{1}{(1+z)^2} \cdot \frac{dx\,dz}{\sqrt{1-x^2-z^2}} = \ldots$$

Slično i za S_2'

(#) Izračunati površinski integral drugog tipa (po koordinatama) $I=\iint\limits_{\sigma}\sqrt[4]{x^2+y^2}\,dx\,dy$ gdje je

σ - donja strana kruga $x^2+y^2\le a^2$.

Rj: Skicirajmo datu površinu



Prisjetimo se, kako se računa površinski integral drugog tipa, npr.

$$\iint\limits_{S} R(x,y,z)\,dx\,dy$$

posmatramo vektor normale $\vec{n}$ na površ S

ako je $\cos\gamma<0$ gdje je γ ugao između $\vec{n}$ i z-ose naš integral postaje

$$\iint\limits_{S} R(x,y,z)\,dx\,dy=-\iint\limits_{D} R(x,y,z(x,y))\,dx\,dy$$

gdje je D ortogonalna projekcija površi S a $z=z(x,y)$ jednačina površi S

U našem slučaju ortogonalna projekcija D je jednaka datoj površini σ. Ugao γ je $\gamma=\pi$ (donja strana kruga) tj. $\cos\pi<0$.

$$I=\iint\limits_{\sigma}\sqrt[4]{x^2+y^2}\,dx\,dy=-\iint\limits_{D}\sqrt[4]{x^2+y^2}\,dx\,dy=$$

uvodimo polarne koordinate: $x=r\cos\varphi$, $y=r\sin\varphi$, $dx\,dy=r\,dr\,d\varphi$, $D\xrightarrow{\text{transf.}}D': \begin{cases}0\le r\le a\\ 0\le\varphi\le 2\pi\end{cases}$

$$=-\iint\limits_{D'}\sqrt[4]{r^2}\,r\,dr\,d\varphi=-\int_0^{2\pi}d\varphi\int_0^a r^{\frac{3}{2}}\,dr=-\int_0^{2\pi}\frac{2}{5}r^{\frac{5}{2}}\Big|_0^a\,d\varphi=-\frac{2}{5}a^{\frac{5}{2}}\varphi\Big|_0^{2\pi}.$$

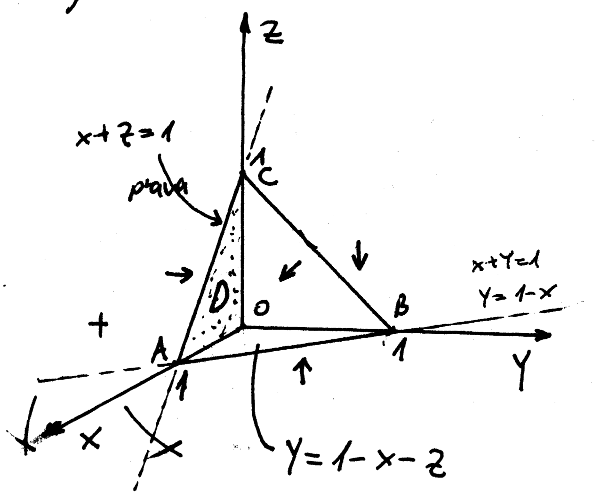
$$I=-\frac{4}{5}\pi\sqrt{a^5}$$ traženo rješenje

(7) Izračunati površinski integral $K=\oiint_{-W} y\,dx\,dz$ gdje je

W - površina tetraedra ograničenog ravnima $x+y+z=1$, $x=0$, $y=0$ i $z=0$.

Rj. integral oblika $\iint_W R(x,y,z)\,dx\,dz$ zovemo površinski integral drugog tipa. Računamo ga tako što napravimo projekciju D površi W na xOz ravan i odredimo predznak broja $\cos\beta$ gdje je β ugao koji zaklapa vektor normale $\vec{n}$ površi W sa Y-osom.

Skicirajmo naš tetraedar



Kako je u zadatku data oblast $-W$ to posmatramo vektore normale koje odgovaraju unutrašnjim površinama tetraedra

$$K=\oiint_{-W} y\,dx\,dz = \iint_{-\Delta AOC} y\,dx\,dz + \iint_{-\Delta AOB} y\,dx\,dz + \iint_{-\Delta BOC} y\,dx\,dz + \iint_{-\Delta ABC} y\,dx\,dz$$

$$\iint_{-\Delta AOC} y\,dx\,dz = +\iint_D 0\cdot dx\,dz = 0$$

$$\iint_{-\Delta AOB} y\,dx\,dz = \left| \begin{matrix} \text{vektor normale } \Delta AOB \\ \text{je okomit na } y\text{-osu} \end{matrix} \right| = 0$$

$$\iint_{-\Delta OBC} y\,dx\,dz = \left| \begin{matrix} \text{vektor normale } \Delta OBC \\ \text{je okomit na } y\text{-osu} \end{matrix} \right| = 0$$

$$\iint_{-\Delta ABC} y\,dx\,dz = \left| \begin{array}{l} \text{vektor normale } \vec{n}^0 \text{ na} \\ \Delta ABC \text{ sa } Y\text{-osom zaklapa} \\ \text{ugao } \beta \text{ koji je između } 90^\circ \text{ i } 180^\circ \\ \text{ZAŠTO? (vidi sliku)} \\ \cos\beta < 0 \end{array} \right| = -\iint_D (1-x-z)\,dx\,dz =$$

$$= -\int_0^1 dx \int_0^{1-x} (1-x-z)\,dz = -\int_0^1 \left(z\Big|_0^{1-x} - xz\Big|_0^{1-x} - \frac{1}{2}z^2\Big|_0^{1-x} \right) dx =$$

$$= -\int_0^1 \left(1-x - x(1-x) - \frac{1}{2}\underbrace{(1-x)^2}_{1-2x+x^2} \right) dx = -\int_0^1 \left(1-x-x+x^2-\frac{1}{2}+x-\frac{1}{2}x^2 \right) dx$$

$$= -\int_0^1 \left(\frac{1}{2}x^2 - x + \frac{1}{2} \right) dx = -\left(\frac{1}{2}\cdot\frac{1}{3}x^3\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1 + \frac{1}{2}x\Big|_0^1 \right) = -\left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2} \right) = -\frac{1}{6}$$

traženo rješenje

II način

Možemo upotrebiti formulu Gauss-Ostrogradski

$$\iint_S P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\,dy\,dz$$

Ω - oblast koju ograničava površ S

U našem slučaju $P(x,y,z) = R(x,y,z) = 0$

$$Q(x,y,z) = y \Rightarrow \frac{\partial Q}{\partial y} = 1$$

$$K = \oiint_{-W} y\,dx\,dz = -\iiint_\Omega dx\,dy\,dz = -\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz = -\int_0^1 dx \int_0^{1-x} (1-x-y)\,dy$$

$$= \left| \begin{array}{l} \text{primjetimo da smo sličan} \\ \text{integral već imali u prethodnom} \\ \text{slučaju} \end{array} \right| = \ldots = -\frac{1}{6}$$

traženo rješenje

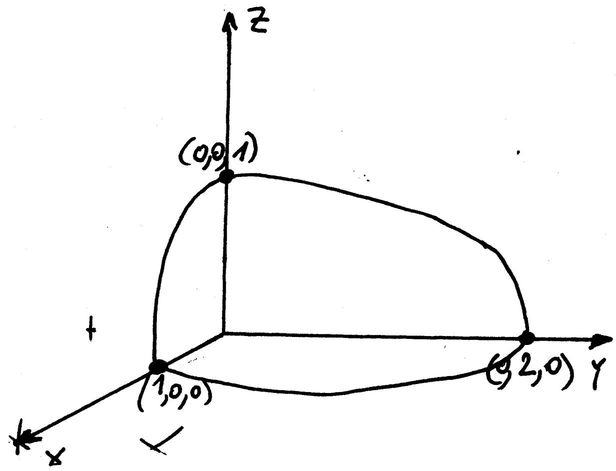
(#) Izračunati površinski integral

$$J=\iint\limits_T 2\,dx\,dy + y\,dx\,dz - x^2 z\,dy\,dz$$ gdje je T vanjska strana elipsoida $4x^2+y^2+4z^2=4$ koji se nalazi u prvom oktantu.

Rj. skicirajmo elipsoid

$4x^2+y^2+4z^2=4 \quad 1:4$

$$\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{1}=1$$



$$J=\iint\limits_T 2\,dx\,dy + y\,dx\,dz - x^2 z\,dy\,dz$$

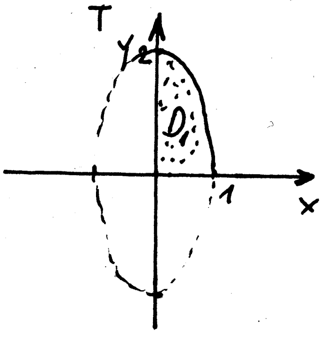
Ovo je površinski integral druge vrste. Prisjetimo se kako se računa npr. $\iint\limits_T P(x,y,z)\,dy\,dz$. Neka je $\vec{n}$ vektor normale površi T koji sa x, y i z osom redom zaklapa uglove α, β i γ, i neka je D ortogonalna projekcija površi T na yoz ravan. Tada

$$\iint\limits_T P(x,y,z)\,dy\,dz = \pm\iint\limits_D P(\eta(y,z),y,z)\,dy\,dz$$ gdje je + ako je $\cos\alpha>0$, − (minus) ako je $\cos\alpha<0$, a $x=\eta(y,z)$ je jednačina koja opisuje površ T.

$$J=\iint\limits_T 2\,dx\,dy + y\,dx\,dz - x^2 z\,dy\,dz = \iint\limits_T 2\,dx\,dy + \iint\limits_T y\,dx\,dz - \iint\limits_T x^2 z\,dy\,dz =$$
$$= J_1 + J_2 - J_3$$

Izračunajmo redom J_1, J_2 i J_3.

$J_1=\iint\limits_T 2\,dx\,dy$; vektor normale $\vec{n}$ na T sa z osom zaklapa ugao $\gamma\in(0,\frac{\pi}{2})$ tj. $\cos\gamma>0$



$z=0: \quad 4x^2+y^2=4$

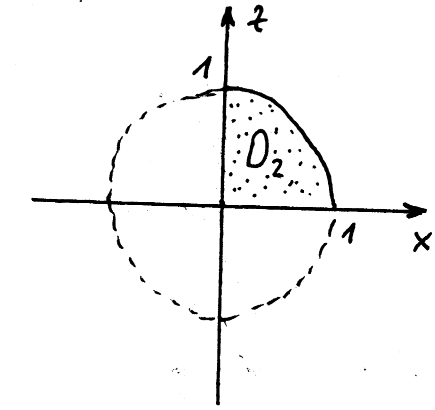
$$D_1:\begin{cases}0\le x\le 1\\ 0\le y\le 2\sqrt{1-x^2}\end{cases}$$

D_1 je četvrtina elipse

$P_{elipse}=ab\pi$, $\quad J_1=+2\iint\limits_{D_1} dx\,dy = 2\cdot\frac{1}{4}P_{elipse}=\frac{1}{2}\cdot 2\pi=\pi$

$J_2 = \iint\limits_T y\,dx\,dz$, vektor normale $\vec{n}$ na površ T sa y-osom zaklapa uglove od 0 do $\pi/2$ (I oktant) pa je $\cos\gamma > 0$.

Neka je D_2 ortogonalna projekcija površi T na xOz ravan.



$D_2: \; 4x^2 + 4z^2 = 4$

$4x^2 + y^2 + 4z^2 = 4$

$y^2 = 4 - 4x^2 - 4z^2$

$y = 2\sqrt{1-x^2-z^2}$

$D_2: \begin{cases} 0 \le x \le 1 \\ 0 \le z \le \sqrt{1-x^2} \end{cases}$

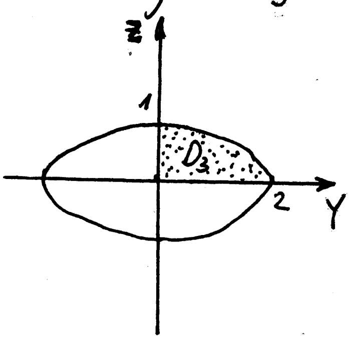
$$J_2 = \iint\limits_T y\,dx\,dz = +2\iint\limits_{D_2} \sqrt{1-x^2-z^2}\,dx\,dz = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x = r\cos\varphi \\ z = r\sin\varphi \\ dz\,dy = r\,dr\,d\varphi \\ D_2 \to D_2' \end{array} \quad D_2': \begin{cases} 0\le r\le 1 \\ 0\le\varphi\le\frac{\pi}{2} \end{cases} \right|$$

$$= 2\iint\limits_{D_2'} \sqrt{1-r^2\cos^2\varphi - r^2\sin^2\varphi}\; r\,dr\,d\varphi = 2\int_0^{\pi/2} d\varphi \int_0^1 \sqrt{1-r^2}\; r\,dr =$$

$$= 2\int_0^{\pi/2} d\varphi \int_0^1 \sqrt{1-r^2}\left(-\frac{1}{2}\right) d(1-r^2) = -\varphi\Big|_0^{\frac{\pi}{2}} \cdot \frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1 = -\frac{\pi}{2}\cdot\left(0-\frac{2}{3}\right) = \frac{\pi}{3}$$

$J_3 = \iint\limits_T x^2 z\,dy\,dz$, vektor normale $\vec{n}$ na površ T sa x-osom zaklapa uglove od 0 do $\frac{\pi}{2}$ pa je $\cos\alpha > 0$

Neka je D_3 ortogonalna projekcija površi T na yOz ravan.



$D_3: \; y^2 + 4z^2 = 4$ $\qquad y^2 = 4 - 4z^2$

$\frac{y^2}{4} + \frac{z^2}{1} = 1$

$4x^2 + y^2 + 4z^2 = 4$

$4x^2 = 4 - y^2 - 4z^2$

$x^2 = 1 - \frac{1}{4}y^2 - z^2$

$$J_3 = \iint\limits_T x^2 z\,dy\,dz = +\iint\limits_{D_3} \left(1 - \frac{1}{4}y^2 - z^2\right) z\,dy\,dz =$$

$$\left| D_3: \begin{cases} 0 \le z \le 1 \\ 0 \le y \le 2\sqrt{1-z^2} \end{cases} \right| = \int_0^1 z\,dz \int_0^{2\sqrt{1-z^2}} \left(1 - \frac{1}{4}y^2 - z^2\right) dy = \int_0^1 z\left(y\Big|_0^{2\sqrt{1-z^2}} - \frac{1}{4}\cdot\frac{1}{3}y^3\Big|_0^{2\sqrt{1-z^2}} - z^2 y\Big|_0^{2\sqrt{1-z^2}}\right)$$

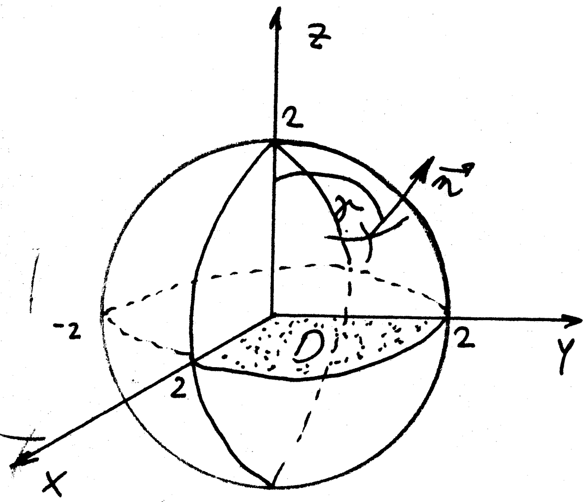
(izvučemo $2\sqrt{1-z^2}$)

$$= \int_0^1 z\left(2\sqrt{1-z^2} - \frac{2}{3}\sqrt{(1-z^2)^3} - 2z^2\sqrt{1-z^2}\right) dz = \frac{4}{3}\int_0^1 z(1-z^2)^{\frac{3}{2}} dz = \frac{2}{3}\cdot\frac{2(1-z^2)^{\frac{5}{2}}}{5}\Bigg|_1^0 = \frac{4}{15}$$

Prema tome $J = \pi + \frac{\pi}{3} - \frac{4}{15} = \frac{4\pi}{3} - \frac{4}{15}$.

Izračunati površinski integral $I = \iint_S x y^3 z \, dx\,dy$, ako je S vanjska strana sfere $x^2+y^2+z^2=4$ u prvom oktantu.

Rj. $x^2+y^2+z^2=4$ je sfera sa centrom u koordinatnom početku čiji je poluprečnik dužine 2.



Kad računamo $\iint_S f(x,y,z)\,dx\,dy$ treba uzeti u obzir predznak broja $\cos\gamma$.
Ako je $\cos\gamma < 0$ ispred integrala stavljamo minus, ako je $\cos\gamma > 0$ ispred integrala stavljamo plus, a ako je $\cos\gamma = 0$ tada je integral jednak 0.
γ je ugao koji vektor normale $\vec{n}$ ($\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$) zaklapa sa z-osom

Vektor normale $\vec{n}$ je u prvom oktantu $\Rightarrow$ $0 < \gamma < \frac{\pi}{2}$

$\Rightarrow \cos\gamma > 0$

$$x^2+y^2+z^2=4$$

$$z = \pm\sqrt{4-(x^2+y^2)} \quad \text{nama treba } +$$

$$I = \iint_S x y^3 z \, dx\,dy = \iint_D x y^3 \left(\sqrt{4-(x^2+y^2)}\right) dx\,dy = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x = r\cos\varphi \\ y = r\sin\varphi \\ dx\,dy = r\,dr\,d\varphi \\ x^2+y^2 = r^2 \end{array} \quad D' : \begin{cases} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2 \end{cases} \right|$$

$$= \iint_{D'} r\cos\varphi \, r^3 \sin^3\varphi \sqrt{4-r^2}\, r\,dr\,d\varphi = \int_0^{\frac{\pi}{2}} \cos\varphi \sin^3\varphi \, d\varphi \int_0^2 r^5\sqrt{4-r^2}\,dr = I_1 \cdot I_2$$

$$I_1 = \int_0^{\pi/2} \cos\varphi \cdot \sin^3\varphi \, d\varphi = \left| \begin{array}{l} \sin\varphi = t \\ \cos\varphi\,d\varphi = dt \\ \varphi\big|_0^{\pi/2} \Rightarrow t\big|_0^1 \end{array} \right| = \int_0^1 t^3 dt = \frac{1}{4} t^4 \Big|_0^1 = \frac{1}{4}$$

$$I_2 = \int_0^2 r^5 \sqrt{4-r^2}\,dr = \int_0^2 r^4 \sqrt{4-r^2}\, r\,dr = \left| \begin{array}{l} 4-r^2 = t^2 \\ -2r\,dr = 2t\,dt \\ r\,dr = -t\,dt \end{array} \quad r\big|_0^2 \Rightarrow t\big|_2^0 \right| = \int_0^2 (4-t^2)^2 \cdot t\,t\,dt$$

$$= \int_0^2 (16-8t^2+t^4)\cdot t^2\,dt = \int_0^2 (t^6-8t^4+16t^2)\,dt = \ldots = \frac{1024}{105} \qquad I = \frac{1}{4}\cdot\frac{1024}{105} = \frac{256}{105}$$

traženo rješenje

(#) Izračunati površinski integral druge vrste

$$I = \iint_S xyz \, dx dy$$

gdje je S spoljna strana dijela sfere $x^2+y^2+z^2=1$, $x \geq 0$, $y \geq 0$.

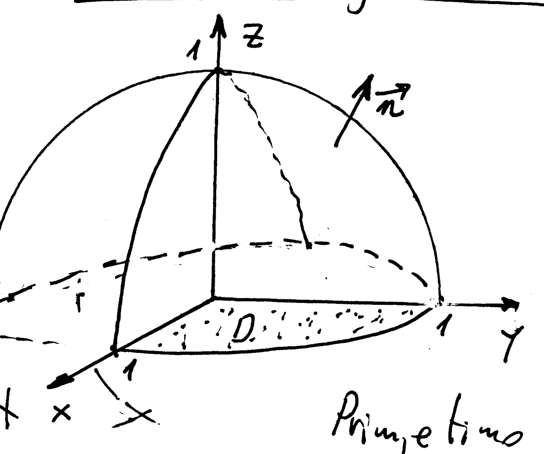
Rj. Prisjetimo se: Neka je površ S data u obliku $z=\eta(x,y)$. Tada

$$\iint_S R(x,y,z)\, dx dy = \pm \iint_D R(x,y,\eta(x,y))\, dx dy$$ gdje

- $\pm$ zavisi od ugla koji vektor normale zaklapa sa z-osom, npr. $\vec{n}_0 = (\cos\alpha, \cos\beta, \cos\gamma)$,

$\cos\gamma > 0$	$\Rightarrow$	$+$
$\cos\gamma < 0$	$\Rightarrow$	$-$
$\cos\gamma = 0$	$\Rightarrow$	0

- D je ortogonalna projekcija površi S na xOy ravan



$$z^2 = 1-x^2-y^2$$

$$z = \pm\sqrt{1-x^2-y^2}$$

kako je $x \geq 0$, $y \geq 0$ to je $z = \sqrt{1-x^2-y^2}$

Primjetimo da je $0 < \gamma < 90°$ $\Rightarrow$ $\cos\gamma > 0$

$$\iint_S xyz\, dx dy = \iint_D xy\sqrt{1-x^2-y^2}\, dx dy = $$

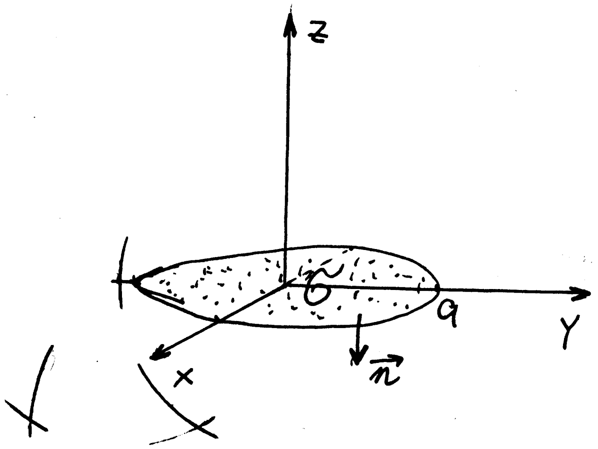
| uvedimo polarne koordinate
$x = r\cos\varphi$, $y = r\sin\varphi$, $dx dy = r\, dr d\varphi$
$1-x^2-y^2 = 1-r^2$
$D \xrightarrow{\text{transf.}} D'$: $\begin{cases} 0 \leq r \leq 1 \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases}$ |

$$= \iint_{D'} r^3 \sin\varphi\cos\varphi\sqrt{1-r^2}\, dr d\varphi = \int_0^1 r^3\sqrt{1-r^2}\, dr \int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi = \underset{\text{za vježbu}}{\ldots} = \frac{2}{15}\cdot\frac{1}{2} = \frac{1}{15}$$

(#) Izračunati površinski integral drugog tipa (po koordinatama) $I = \iint\limits_{\sigma} \sqrt[4]{x^2+y^2}\, dx\, dy$ gdje je

σ - donja strana kruga $x^2+y^2 \le a^2$.

Rj: Skicirajmo datu površinu



Prisjetimo se, kako se računa površinski integral drugog tipa, npr.

$$\iint\limits_{S} R(x,y,z)\, dx\, dy$$

posmatramo vektor normale $\vec{n}$ na površ S

ako je $\cos\gamma < 0$ gdje je γ ugao između $\vec{n}$ i z-ose naš integral postaje

$$\iint\limits_{S} R(x,y,z)\, dx\, dy = -\iint\limits_{D} R(x,y,z(x,y))\, dx\, dy$$

gdje je D ortogonalna projekcija površi S a $z=z(x,y)$ jednačina površi S

U našem slučaju ortogonalna projekcija D je jednaka datoj površini σ. Ugao γ je $\gamma = \pi$ (donja strana kruga) tj. $\cos\pi < 0$.

$$I = \iint\limits_{\sigma} \sqrt[4]{x^2+y^2}\, dx\, dy = -\iint\limits_{D} \sqrt[4]{x^2+y^2}\, dx\, dy =$$

uvodimo polarne koordinate
$x = r\cos\varphi$
$y = r\sin\varphi$
$dx\, dy = r\, dr\, d\varphi$
$D \xrightarrow{\text{transf.}} D' : \begin{cases} 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases}$

$$= -\iint\limits_{D'} \sqrt[4]{r^2}\, r\, dr\, d\varphi = -\int_0^{2\pi} d\varphi \int_0^a r^{\frac{3}{2}} dr = -\int_0^{2\pi} \frac{2}{5} r^{\frac{5}{2}} \Big|_0^a d\varphi = -\frac{2}{5} a^{\frac{5}{2}} \varphi \Big|_0^{2\pi}.$$

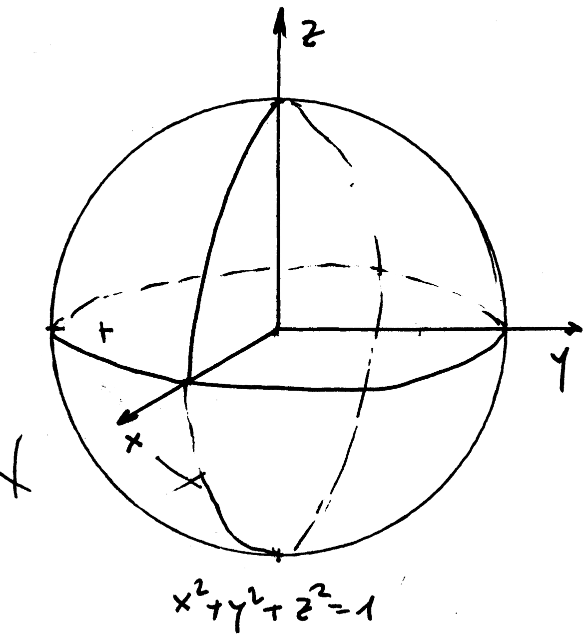
$$I = -\frac{4}{5}\pi\sqrt{a^5}$$ traženo rješenje

Izračunati $I = \iint\limits_{S+} \left(\frac{1}{x} dy dz + \frac{1}{y} dz dx + \frac{1}{z} dx dy \right)$

gdje je $S+$ spoljašnja strana jedinične sfere (zadatak uraditi bez upotrebe teoreme Gauss-Ostrogradskog – zadatak se ne može uraditi uz pomoć navedene teoreme zato što zadatak ne ispunjava sve uslove teoreme)

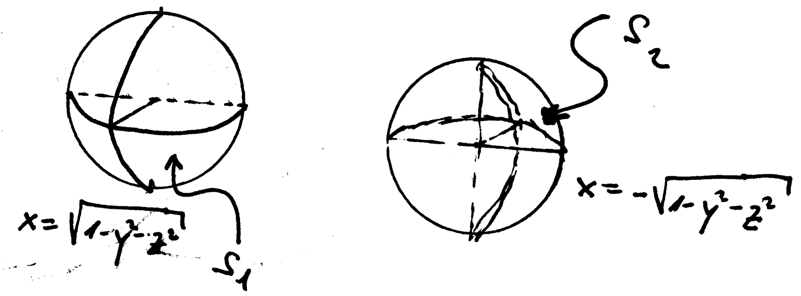
Rj. Podijelimo dati integral na tri dijela:

$$I = \iint\limits_{S+} \frac{1}{x} dy dz + \iint\limits_{S+} \frac{1}{y} dz dx + \iint\limits_{S+} \frac{1}{z} dx dy = I_1 + I_2 + I_3$$



Posmatrajmo $I_1 = \iint\limits_{S+} \frac{1}{x} dy dz$

Zbog vektora normale, u ovom slučaju, spoljašnju stranu jedinične sfere trebamo podijeliti na dva dijela S_1 i S_2



$$I_1 = \iint\limits_{S+} \frac{1}{x} dy dz = \iint\limits_{S_1} \frac{1}{x} dy dz + \iint\limits_{S_2} \frac{1}{x} dy dz$$

$$\iint\limits_{S_1} \frac{1}{x} dy dz = \left| \begin{array}{l} \text{• ugao } \alpha \text{ između vektora normale } \vec{n} \\ \text{i x-ose je između 0 i } \pi/2 \\ 0 \le \cos\alpha \le 1 \quad \cos\alpha \ge 0 \\ \text{• } x = \sqrt{1-y^2-z^2} \\ \text{• ortogonalna projekcija od } S_1 \\ \text{na xOy ravan je krug } y^2+z^2=1 \end{array} \right| = + \iint\limits_{D} \frac{dy dz}{\sqrt{1-y^2-z^2}} =$$

$$= \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ y = \rho\cos\varphi \\ z = \rho\sin\varphi \\ dy\,dz = \rho\, d\rho\, d\varphi \end{array} \quad D \xrightarrow{\text{transf.}} D': \begin{cases} 0 \le \rho \le 1 \\ 0 \le \varphi \le 2\pi \end{cases} \right| = \iint_{D'} \frac{\rho\, d\rho\, d\varphi}{\sqrt{1-\rho^2}} = \int_0^1 \frac{\rho\, d\rho}{\sqrt{1-\rho^2}} \int_0^{2\pi} d\varphi =$$

$$= \left| \begin{array}{l} d(1-\rho^2) = -2\rho\, d\rho \\ \rho\, d\rho = -\frac{1}{2} d(1-\rho^2) \end{array} \right| = 2\pi \int_0^1 \frac{-\frac{1}{2} d(1-\rho^2)}{\sqrt{1-\rho^2}} = -\pi \cdot \frac{(1-\rho^2)^{\frac{1}{2}}}{\frac{1}{2}} \Big|_0^1 =$$

$$= -2\pi(0-1) = 2\pi$$

$$\iint_{S_2} \frac{1}{x}\, dy\, dz = \left| \begin{array}{l} \bullet\ x = -\sqrt{1-y^2-z^2} \\ \bullet\ \alpha \in (\pi/2, \pi) \Rightarrow \cos\alpha \le 0 \\ \bullet\ \text{ortog. proj. je } D: y^2+z^2=1 \end{array} \right| = -\iint_D \frac{dy\, dz}{-\sqrt{1-y^2-z^2}} = \iint_{S_1} \frac{1}{x}\, dy\, dz$$

$$= 2\pi$$

Primjetimo da ista priča važi za $I_2 = \iint_{S+} \frac{1}{y}\, dx\, dz$, $I_3 = \iint_{S+} \frac{1}{z}\, dx\, dy$ (u oba slučaja $S+$ se podjeli na dvije polusfere i čitav račun bude sličan računu iznad). Prema tome

$$I = 4\pi + 4\pi + 4\pi = 12\pi \quad \text{traženo rješenje.}$$

Izračunati površinski integral

$$I = \iint\limits_{S+} y^2 dy dz + (y^2 + x^2) dz dx + (y^2 + x^2 + z^2) dx dy,$$

gdje je $S+$ spoljašnja strana polusfere $x^2 + y^2 + z^2 = 2Rx$, $z > 0$ (za fiksirano $R > 0$).

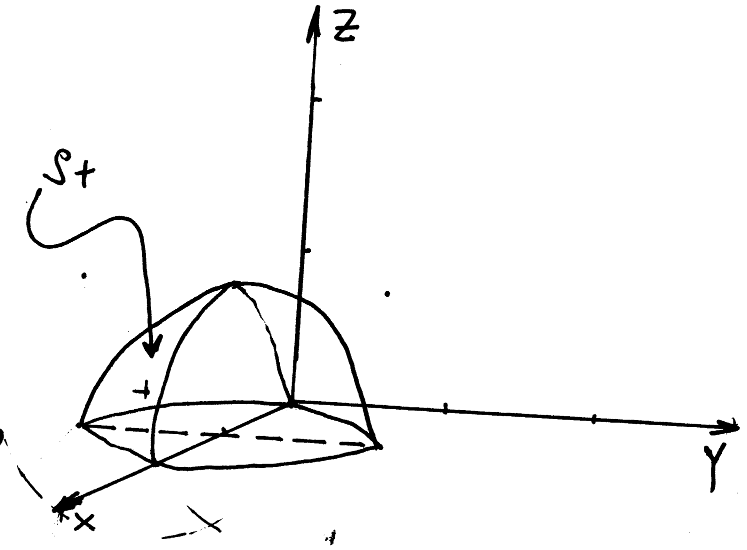
Rj.

$$x^2 + y^2 + z^2 = 2Rx$$

$$x^2 - 2 \cdot x \cdot R + R^2 - R^2 + y^2 + z^2 = 0$$

$$(x - R)^2 + y^2 + z^2 = R^2$$

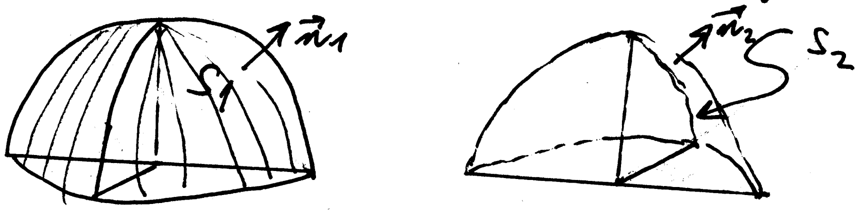
ovo je sfera sa centrom u tački $(R, 0, 0)$ poluprečnika R



Dati integral podjelimo na tri dijela

$$I = \iint\limits_{S+} y^2 dy dz + \iint\limits_{S+} (y^2 + x^2) dz dx + \iint\limits_{S+} (y^2 + x^2 + z^2) dx dy = I_1 + I_2 + I_3$$

Da bi izračunali I_1 treba nam vektor normale na površ $S+$. Sa slike vidimo da površ $S+$ trebamo podjeliti na dva dijela S_1 i S_2



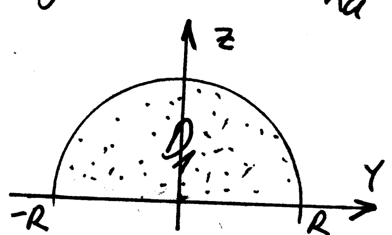
$\measuredangle(\vec{n}_1, x\text{-ose}) \in (0, \pi/2)$

$\Rightarrow \cos\alpha > 0$

$\measuredangle(\vec{n}_2, x\text{-ose}) \in (\pi/2, \pi)$

$\Rightarrow \cos\alpha < 0$

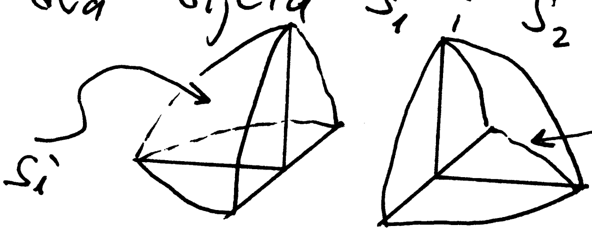
Ortogonalna projekcija sfere na yOz ravan je



$$I_1 = \iint\limits_{S+} y^2\,dy\,dz = \iint\limits_{S_1} y^2\,dy\,dz + \iint\limits_{S_2} y^2\,dy\,dz$$

$$\left.\begin{aligned} \iint\limits_{S_1} y^2\,dy\,dz &= +\iint\limits_{D_1} y^2\,dy\,dz \\ \iint\limits_{S_2} y^2\,dy\,dz &= -\iint\limits_{D_1} y^2\,dy\,dz \end{aligned}\right\} \Rightarrow I_1 = 0$$

Da bi izračunali $I_2 = \iint\limits_{S+} (y^2+x^2)\,dz\,dx$, slično kao za I_1, treba nam ugao β između vektora normale $\vec{n}$ i y-ose. Da bi našli ovaj ugao površ S trebamo podjeliti na dva dijela S_1' i S_2'.



za S_1' $\pi/2 \le \beta \le \pi$

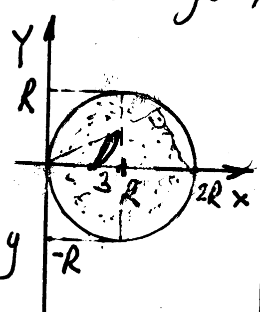
$\cos\beta \le 0$

za S_2' $0 \le \beta \le \pi/2$

$\cos\beta \ge 0$

$\Rightarrow$ $I_2 = 0$ (zato što imamo + i − i ose površi imaju istu ortogonalnu projekciju na xOz ravan).

$$I_3 = \iint\limits_{S+} (x^2+y^2+z^2)\,dx\,dy = \left|\begin{array}{l} \bullet \text{ ugao između vektora normale } \vec{n} \text{ na } S \text{ i } z\text{-ose je između } 0 \text{ i } \pi/2 \Rightarrow \cos\gamma > 0 \\ \bullet \text{ ortogonalna projekcija je krug} \\ \bullet\ z^2 = 2Rx - x^2 - y^2 \end{array}\right|$$



$(x-R)^2 + y^2 = R^2$

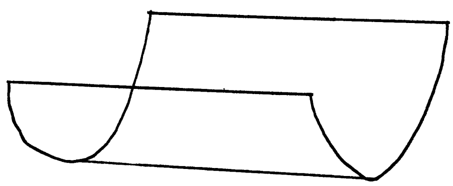
$x - R = \pm\sqrt{R^2-y^2}$

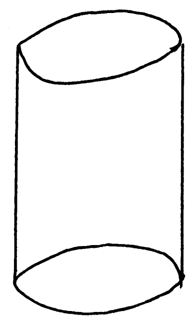
$$= +\iint\limits_{D_3} 2Rx\,dx\,dy = \left|\begin{array}{l} \text{uvedimo polarne koordinate} \\ x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ dx\,dy = \rho\,d\rho\,d\varphi \end{array} \quad D \xrightarrow{\text{transform.}} D': \begin{cases} 0 < \rho \le 2R\cos\varphi \\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \end{cases} \quad \cos\varphi = \frac{\rho}{2R} \right| =$$

$$= 2R \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2R\cos\varphi} \rho\cos\varphi\,\rho\,d\rho \overset{\text{ZA VJEŽBU}}{=} \ldots = \frac{16R^4}{3} \int_{-\pi/2}^{\pi/2} \cos^4\varphi\,d\varphi \overset{\text{ZA VJEŽBU}}{=} 2\pi R^4$$ traženo rješenje

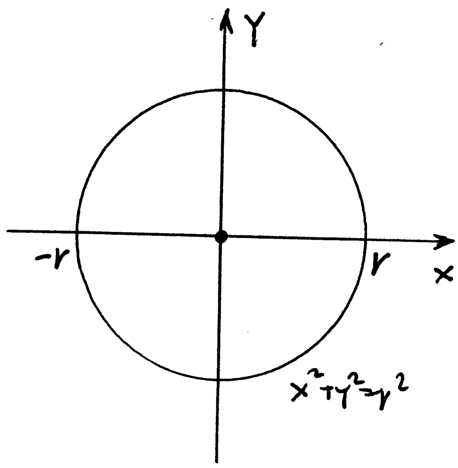
(#) Data je kriva c koja je dobijena kao presjek površina $x^2+y^2=r^2$ i $x^2=rz$ $(r>0)$. Izračunati površinski integral $\iint\limits_S dxdy$ gdje je S gornja strana površine koju zatvara kriva c.

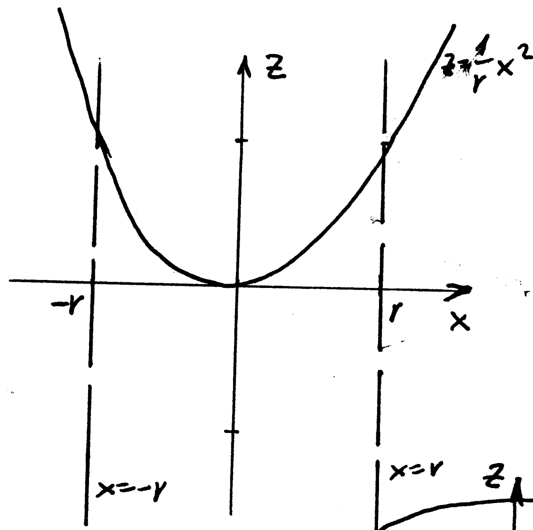
Rj. U prostoru $x^2+y^2=r^2$ predstavlja cilindar, dok $x^2=rz$ predstavlja paraboloid

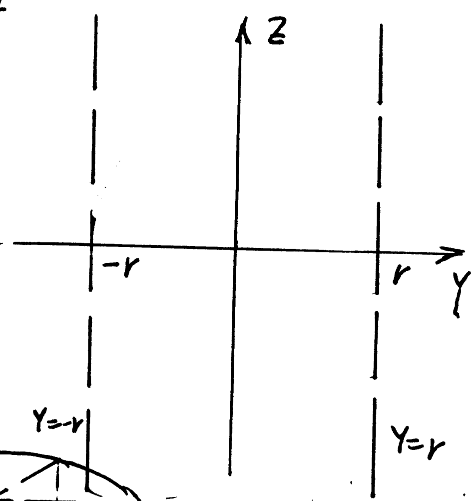




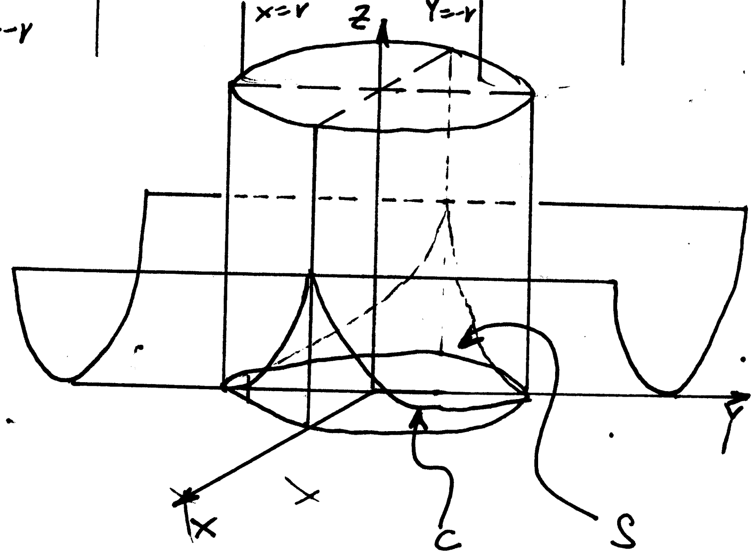
Napravimo presjeke datih površina sa koordinatnim ravnima







Na osnovu presjeka možemo skicirati sliku u prostoru.



$\iint_S dxdy =$ | • ugao između vektora normale $\vec{n}$ na površ S i z-ose je uvijek između 0 i $\frac{\pi}{2}$ pa je $\cos\gamma > 0$
• projekcija od S na xOy ravan je krug $x^2+y^2=r^2$ | $= +\iint_D dxdy =$

$=$ | $D: x^2+y^2=r^2$
uvedimo polarne koordinate
$x=\rho\cos\varphi$
$y=\rho\sin\varphi$
$dxdy=\rho\, d\rho\, d\varphi$

$D \xrightarrow{\text{transformise}} D': \begin{cases} 0 \le \rho \le r \\ 0 \le \varphi \le 2\pi \end{cases}$ | $= \iint_{D'} \rho\, d\rho\, d\varphi$

$$= \int_0^r \rho\, d\rho \int_0^{2\pi} d\varphi = \frac{1}{2}\rho^2\Big|_0^r \cdot \varphi\Big|_0^{2\pi} = \pi r^2$$ traženo rješenje

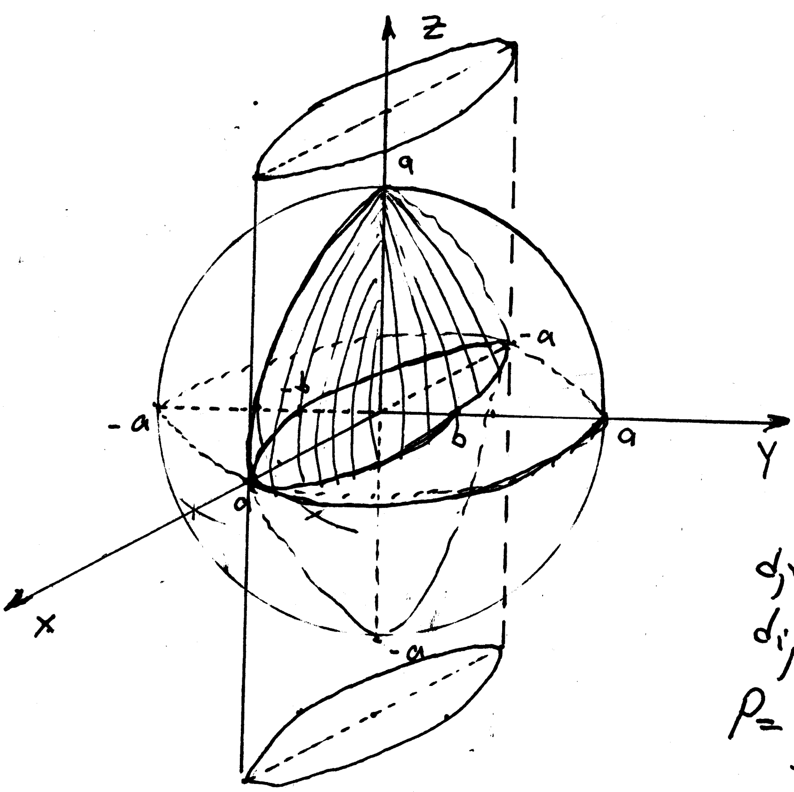
(#) Izračunati površinu dijela sfere

$$S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=a^2\}$$

koji se nalazi u unutrašnjosti cilindra

$$S_1=\{(x,y,z)\in\mathbb{R}^3 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z\in\mathbb{R}\},\quad b\le a$$

R: Skicirajmo sferu S i cilindar S_1.



Cilindrična površ u presjeku sa sferom, isjeca iz nje simetričnu površ u odnosu na ravan xOy. Ta dva simetrična dijela označimo sa l_1 i l_2. Svaki od ova dva dijela, koordinatne ravni xOz i yOz ih dijele na četiri jednaka dijela.

$P=\iint_S dS$ gdje je S površina dijela sfere ograničena cilindrom.

$S:\ x^2+y^2+z^2=a^2$

$$z=\pm\sqrt{a^2-x^2-y^2}$$

Zbog navedene simetričnosti posmatraću sferu samo u prvom oktantu

$$z=\sqrt{a^2-x^2-y^2}\qquad z'_x=\frac{-x}{\sqrt{a^2-x^2-y^2}},\ z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$\iint_S dS=\iint_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$ gdje je D projekcija površi S na xOy ravan

$$1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}=\frac{a^2}{a^2-x^2-y^2}$$

$$P=8\iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\\ a\ge 0,\ b\ge 0}}\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dx\,dy=8a\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ a\ge 0,\ b\ge 0}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}=8a\iint_D\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$

gdje je $D: \begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1 \\ x \ge 0,\ y \ge 0 \end{cases}$ ili drugačije napisano $D: \begin{cases} 0 \le x \le a \\ 0 \le y \le \frac{b}{a}\sqrt{a^2-x^2} \end{cases}$

$$\frac{y^2}{b^2} \le 1-\frac{x^2}{a^2}$$

$$y^2 \le \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

$$P = 8a\int_0^a dx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{dy}{\sqrt{\underbrace{a^2-x^2}_{\text{ovo je broj za } dy}-y^2}} = 8a\int_0^a \left(\arcsin\frac{y}{\sqrt{a^2-x^2}}\Bigg|_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}}\right)dx$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+C$$

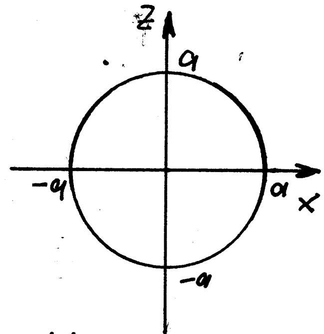
$$= 8a\int_0^a \left(\arcsin\frac{b}{a} - \underbrace{\arcsin 0}_{=0}\right)dx =$$

$$= 8a \arcsin\frac{b}{a}\int_0^a dx = 8a^2 \arcsin\frac{b}{a}$$ tražena površina

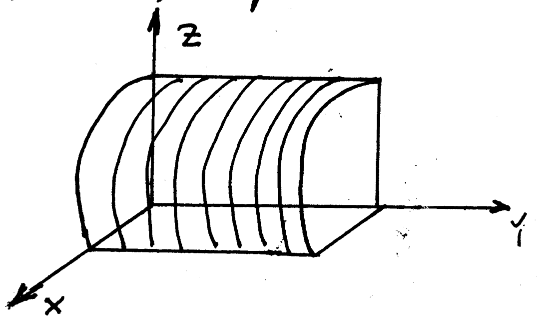
(#) Neka je S površina tijela koje je dobijeno presjekom dva cilindra $S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+z^2=a^2,\ y \in \mathbb{R}\}$ i $S_2 = \{(x,y,z) \in \mathbb{R}^3 \mid y^2+z^2=a^2,\ x \in \mathbb{R}\}$. Izračunati površinu dobijenog tijela.

Rj. $P = \iint\limits_S dS$ Skicirajmo S_1 i S_2, pa skicirajmo njihov presjek.

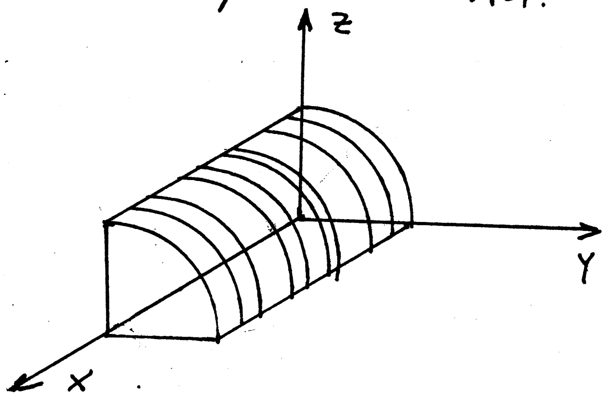
S_1: $x^2+z^2=a^2$ u ravni: xOz



U prostoru, u prvom oktantu:



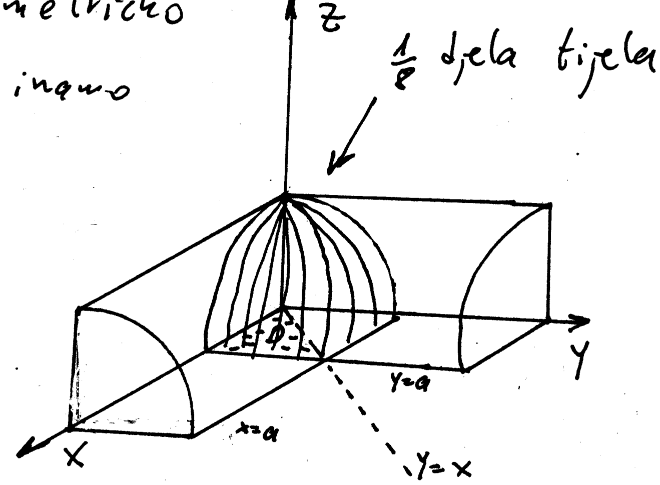
S_2 u prvom oktantu:



Presjek $S_1 \cap S_2$ će kao rezultat dati tijelo koje je simetrično u odnosu na sve tri ravni xOy, xOz i yOz.

$\frac{1}{8}$ dijela tijela će se nalaziti u prvom oktantu:

Primjetimo da je i ovo tijelo simetrično u odnosu na pravu $y=x$ pa imamo



$$P = \frac{1}{16} \iint\limits_D \sqrt{1+z_x'^2+z_y'^2}\, dx\,dy$$

gdje je $D: \begin{cases} 0 \le x \le a \\ 0 \le y \le x \end{cases}$

$z^2 = a^2 - x^2$ tj. $z = \sqrt{a^2-x^2}$.

$z_x' = \frac{-x}{\sqrt{a^2-x^2}}$, $z_y' = 0$ $\quad 1+(z_x')^2+(z_y')^2 = 1+\frac{x^2}{a^2-x^2} = \frac{a^2}{a^2-x^2}$

tražena površina

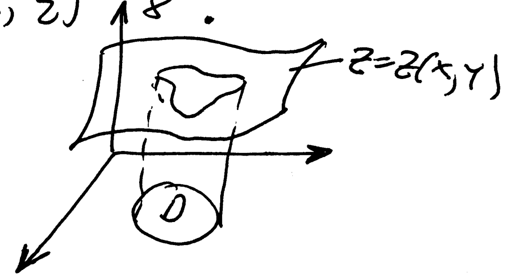
$$P = 16a\int_0^a \frac{dx}{\sqrt{a^2-x^2}} \int_0^x dy = 16a\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}} = \left| \begin{array}{l} a^2-x^2=t \\ -2x\,dx=dt \\ x\,dx=-\frac{1}{2}dt \\ \ldots \end{array} \right| = \ldots = 16a\sqrt{a^2-x^2}\Big|_a^0 = 16a^2$$

(#) Izračunati površinu dijela površi $S: z^2 = 2xy$ određene u prvom oktantu u presjeku sa ravnima: $x=0$, $y=0$ i $x+y=1$.

Uputa: $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$, $B(\frac{3}{2},\frac{3}{2}) = \frac{\pi}{8}$, $B(\frac{1}{2},\frac{5}{2}) = \frac{3\pi}{8}$.

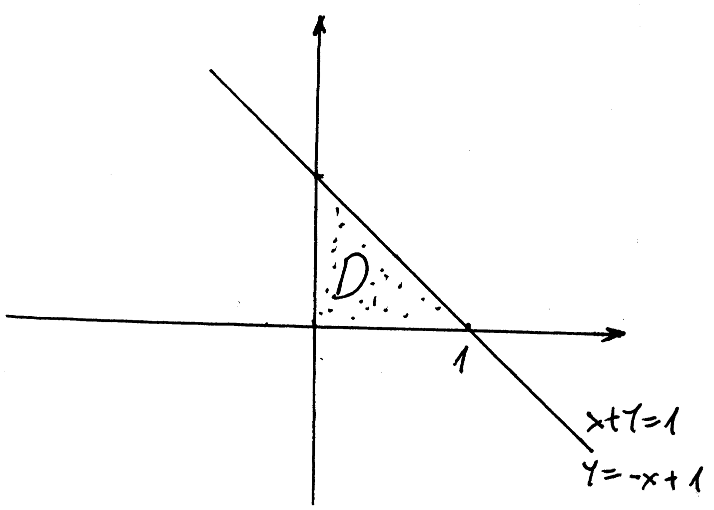
Rj:

$$P = \iint_S dS = \iint_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$



Kako je površina S u prvom oktantu, u našem slučaju je

$S: \quad z = \sqrt{2}\sqrt{xy}$



$$z'_x = \sqrt{2}\,\frac{y}{2\sqrt{xy}}$$

$$z'_y = \sqrt{2}\,\frac{x}{2\sqrt{xy}}$$

$$1+(z'_x)^2+(z'_y)^2 = 1+\frac{y^2}{2xy}+\frac{x^2}{2xy} = \frac{2xy+y^2+x^2}{2xy} = \frac{(x+y)^2}{2xy}$$

$$P = \iint_S dS = \iint_D \frac{x+y}{\sqrt{2xy}}\,dx\,dy = \frac{1}{\sqrt{2}}\int_0^1 dx \int_0^{-x+1} \frac{(x+y)}{\sqrt{xy}}\,dy =$$

$$= \frac{1}{\sqrt{2}}\int_0^1 dx \int_0^{1-x}\left(x\cdot x^{-\frac{1}{2}}\cdot y^{-\frac{1}{2}} + y\cdot x^{-\frac{1}{2}}\cdot y^{-\frac{1}{2}}\right)dy = \frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{1-x}\left(x^{\frac{1}{2}}y^{-\frac{1}{2}} + x^{-\frac{1}{2}}y^{\frac{1}{2}}\right)dy$$

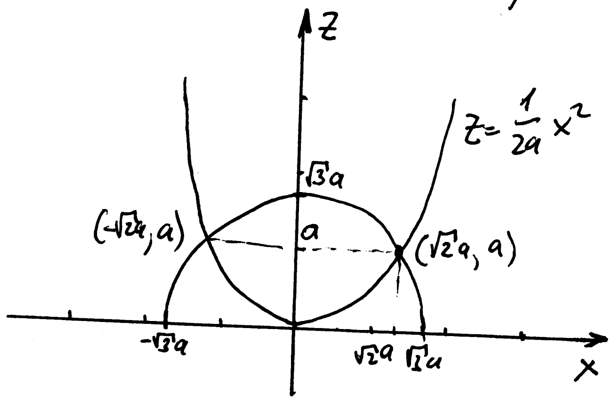
$$= \frac{1}{\sqrt{2}}\int_0^1 \left(x^{\frac{1}{2}}\,\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\Big|_0^{1-x} + x^{-\frac{1}{2}}\cdot\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^{1-x}\right)dx = \frac{2}{\sqrt{2}}\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}dx +$$

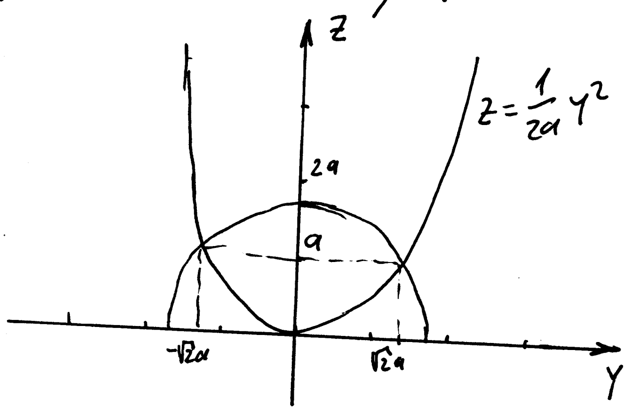
$$+ \frac{2}{3\sqrt{2}}\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}}dx = \sqrt{2}\int_0^1 x^{\frac{3}{2}-1}(1-x)^{\frac{3}{2}}dx + \frac{\sqrt{2}}{3}\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{5}{2}-1}dx = \sqrt{2}\,B\left(\frac{3}{2},\frac{3}{2}\right) +$$

$$+ \frac{\sqrt{2}}{3}B\left(\frac{1}{2},\frac{5}{2}\right) = \frac{\pi}{2\sqrt{2}}.$$

Izračunati površinu dijela lopte $x^2+y^2+z^2=3a^2$ koja se nalazi ispod parabole $x^2+y^2=2az$ a iznad xOy ravni.

Rj: Na osnovu skica presjeka datih površina sa xOz i yOz ravnima ćemo vidjeti kaka tijela su u pitanju.





$$x^2+z^2=3a^2$$
$$x^2=2az$$

$$z^2+2az-3a^2=0$$
$$D=4a^2+12a^2=16a^2$$
$$z_{1,2}=\frac{-2a\pm 4a}{2}$$
$$z_1=a \quad z_2=-3a$$

$P=\iint\limits_S dS$ površinski integral prve vrste

$$z^2=3a^2-x^2-y^2$$
$$z=\pm\sqrt{3a^2-x^2-y^2}$$

U našem slučaju S je $z=\sqrt{3a^2-x^2-y^2}$ i to do one površine koji se nalazi ispod parabole

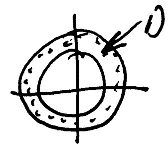
$$P=\iint\limits_S dS=\iint\limits_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$z'_x=\frac{-2x}{2\sqrt{3a^2-x^2-y^2}}, \quad z'_y=\frac{-y}{\sqrt{3a^2-x^2-y^2}}$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{3a^2-x^2-y^2}+\frac{y^2}{3a^2-x^2-y^2}=\frac{3a^2}{3a^2-x^2-y^2}$$

$$P=\sqrt{3}a\iint\limits_D\frac{dx\,dy}{\sqrt{3a^2-x^2-y^2}}$$

gdje je D projekcija površi S na xOy ravan. U našem slučaju



Uvedimo polarne koordinate

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$dx\,dy = r\,dr\,d\varphi$$

$$D \xrightarrow{\text{transformise}} D': \begin{cases} \sqrt{2}a \le r \le \sqrt{3}a \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$x^2 + y^2 = r^2$$

$$P = \sqrt{3}a \iint\limits_{D'} \frac{r\,dr\,d\varphi}{\sqrt{3a^2 - r^2}} = \sqrt{3}a \int_0^{2\pi} d\varphi \int_{\sqrt{2}a}^{\sqrt{3}a} \frac{r\,dr}{\sqrt{3a^2 - r^2}} = \left| \begin{array}{l} 3a^2 - r^2 = t^2 \\ -2r\,dr = 2t\,dt \\ r\Big|_{\sqrt{2}a}^{\sqrt{3}a} \Rightarrow t\Big|_a^0 \end{array} \right|$$

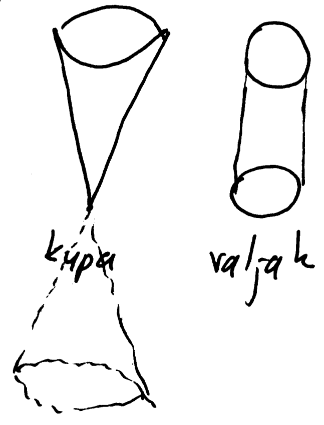
$$= \sqrt{3}a \int_0^{2\pi} d\varphi \int_0^a \frac{t\,dt}{t} = 2a^2\sqrt{3}\pi$$

traženo rešenje

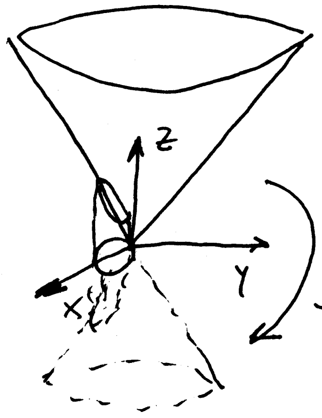
2π a

(#) Izračunati površinu onog dijela kupe $z^2=x^2+y^2$ koji se nalazi unutar valjka $x^2+y^2=2x$.

Rj.



Prema zadatku dio kupe se nalazi unutar valjka



$x^2+y^2=2x$

$x^2-2\cdot x\cdot 1+1+y^2=1$

$(x-1)^2+y^2=1$

sličnu figuru ćemo imati i sa druge strane xOy ravni

$P=\iint_S dS$ gdje je S dio kupe koji se nalazi unutar valjka

$$P=\iint_D \sqrt{1+z_x'^2+z_y'^2}\,dx\,dy$$

$z=\pm\sqrt{x^2+y^2}$

Ako za z uzmemo $z=\sqrt{x^2+y^2}$ dobićemo površinu dijela kupe iznad xOy ravni.

$z=\sqrt{x^2+y^2}$

$$z_x'=\frac{x}{\sqrt{x^2+y^2}},\quad z_y'=\frac{y}{\sqrt{x^2+y^2}},\quad 1+z_x'^2+z_y'^2=1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}=1+1=2$$

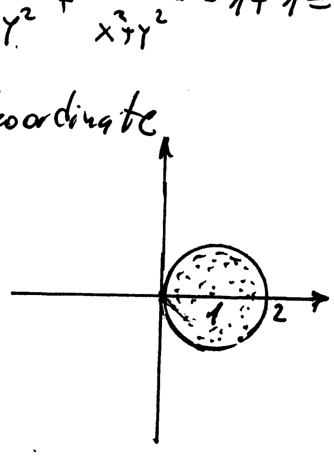
Uvedimo polarne koordinate

$x=r\cos\varphi+1$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

D: unutrašnjost kruga $x^2+y^2=2x$



$$D \xrightarrow{\text{transf.}} D':\begin{cases}0\le r\le 1\\ 0\le\varphi\le 2\pi\end{cases}$$

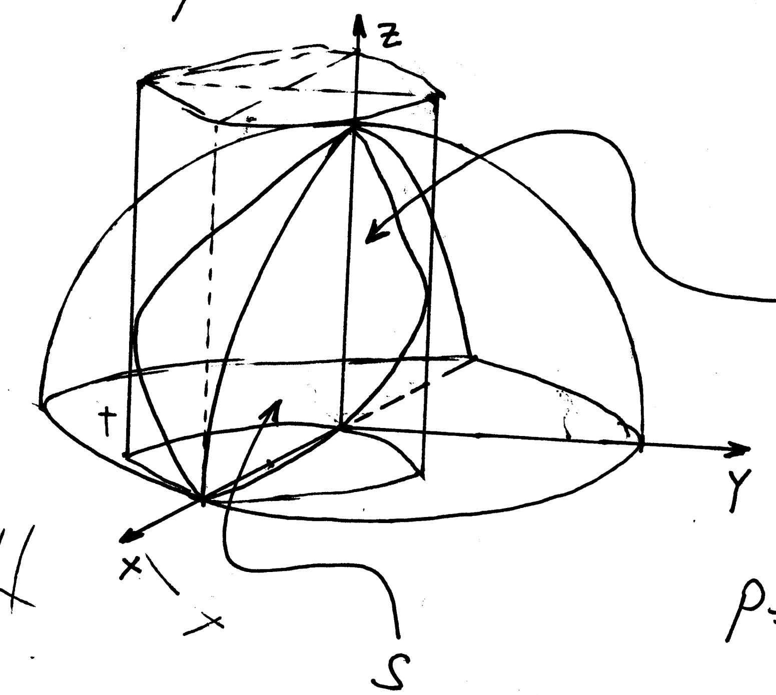
$$\frac{1}{2}P=\iint_D \sqrt{2}\,dx\,dy=\sqrt{2}\iint_{D'} r\,dr\,d\varphi=\sqrt{2}\int_0^{2\pi}d\varphi\int_0^1 r\,dr=\sqrt{2}\cdot\frac{1}{2}r^2\Big|_0^1\varphi\Big|_0^{2\pi}=\sqrt{2}\pi$$

$$P=2\sqrt{2}\pi$$

(#) Odrediti površinu koju cilindar $x^2+y^2=ax$ isjeca na lopti $x^2+y^2+z^2=a^2$ iznad ravni Oxy.

Rj.

Skicirajmo sliku



$$x^2+y^2=ax$$

$$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

Trebamo izračunati površinu dijela lopte koji se nalazi unutar cilindra.

$$P=\iint_S dS=\iint_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$z^2=a^2-x^2-y^2$$

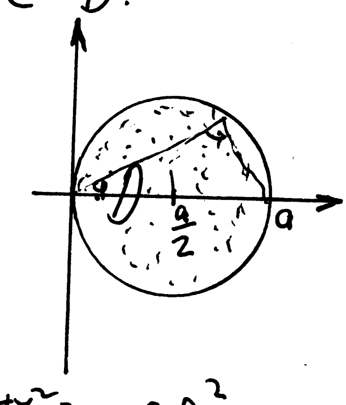
$$z=\pm\sqrt{a^2-x^2-y^2}$$

kako je u pitanju gornji dio polusfere to imamo

$$z=+\sqrt{a^2-x^2-y^2}$$

$$z'_x=\frac{-x}{\sqrt{a^2-x^2-y^2}},\quad z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}},\quad 1+z'^2_x+z'^2_y=\frac{a^2-x^2-y^2+x^2+y^2}{a^2-x^2-y^2}$$

$$P=\iint_D\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dx\,dy=a\iint_D\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$ gdje je D:



Uvedimo polarne koordinate

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transformiše}} D'$: $\begin{cases}0\le\rho\le a\cos\varphi\\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$

$\cos\varphi=\frac{\rho}{a}$

$x^2+y^2=\dots=\rho^2$

$$a\iint_D \frac{dx\,dy}{\sqrt{a^2-(x^2+y^2)}} = \left| \begin{matrix} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{matrix} \right| = a\iint_{D'} \frac{\rho\,d\rho\,d\varphi}{\sqrt{a^2-\rho^2}}$$

$$= a\int_{-\pi/2}^{\pi/2} d\varphi \int_0^{a\cos\varphi} \frac{\rho\,d\rho}{\sqrt{a^2-\rho^2}} = \left| \begin{matrix} d(a^2-\rho^2) = -2\rho\,d\rho \\ \rho\,d\rho = -\frac{1}{2}\,d(a^2-\rho^2) \end{matrix} \right| =$$

$$= a\int_{-\pi/2}^{\pi/2} d\varphi \int_0^{a\cos\varphi} -\frac{1}{2}\,(a^2-\rho^2)^{-\frac{1}{2}}\,d(a^2-\rho^2) = -\frac{1}{2}\cdot 2a \int_{-\pi/2}^{\pi/2} \underbrace{(a^2-\rho^2)^{\frac{1}{2}}\Big|_0^{a\cos\varphi}}_{(a^2-a^2\cos^2\varphi)^{\frac{1}{2}}-(a^2-0)^{\frac{1}{2}}} d\varphi$$

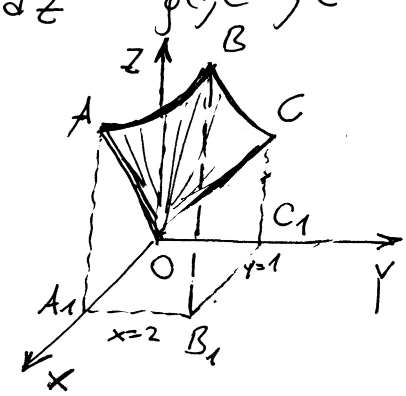
$$(a^2-a^2\cos^2\varphi)^{\frac{1}{2}} = \big(a^2\underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi}\big)^{\frac{1}{2}}$$

$$= -a\int_{-\pi/2}^{\pi/2} (a\sin\varphi - a)\,d\varphi$$

$$= -a^2\int_{-\pi/2}^{\pi/2} (\sin\varphi - 1)\,d\varphi = -a^2\cdot\Big(\underbrace{-\cos\varphi\Big|_{-\pi/2}^{\pi/2}}_{0} - \underbrace{\varphi\Big|_{-\pi/2}^{\pi/2}}_{\frac{\pi}{2}+\frac{\pi}{2}}\Big) = a^2\pi$$

traženo rješenje

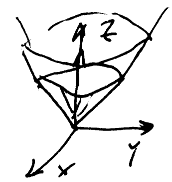
Uz pomoć formule Stoksa, izračunati krivolinijski integral $K=\oint_{\ell} e^x dx + z(x^2+y^2)^{\frac{3}{2}} dy + yz^3 dz$ gdje je



ℓ - zakrivljena linija OCBAO (vidi sliku) dobijena presjekom površina

$z=\sqrt{x^2+y^2}$, $x=0$, $x=2$, $Y=0$, $Y=1$.

Rj.:



$z=\sqrt{x^2+y^2}$ je čunj iznad xOy ravni

$x=0$, $x=2$ su ravni paralelne sa yOz ravni

$y=0$, $y=1$ su ravni paralelne sa xOz ravni

Stoksova formula glasi

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

$P=e^x$, $\frac{\partial P}{\partial y}=0$, $\frac{\partial P}{\partial z}=0$

$Q=z(x^2+y^2)^{\frac{3}{2}}$, $\frac{\partial Q}{\partial x} = z\cdot\frac{3}{2}(x^2+y^2)^{\frac{1}{2}}\cdot 2x = 3xz\sqrt{x^2+y^2}$, $\frac{\partial Q}{\partial z}=(x^2+y^2)^{\frac{3}{2}}$

$R=yz^3$, $\frac{\partial R}{\partial x}=0$, $\frac{\partial R}{\partial y}=z^3$.

$$K=\oint_{\ell} e^x dx + z(x^2+y^2)^{\frac{3}{2}} dy + yz^3 dz = \left| \text{formula Stoksa} \right| =$$

$$= \iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & z(x^2+y^2)^{\frac{3}{2}} & yz^3 \end{vmatrix} = \iint_S \underbrace{\left(z^3-(x^2+y^2)^{\frac{3}{2}}\right)}_{=0} dy\,dz - (0-0)\,dx\,dz$$

(where $z^3 = \left(\sqrt{x^2+y^2}\right)^3$)

$$+\left(3xz\sqrt{x^2+y^2}-0\right)dx\,dy = \iint_S 3xz\sqrt{x^2+y^2}\,dx\,dy$$

površinski integral II vrste

Tj. dobili smo $K=\iint\limits_S 3xz\sqrt{x^2+y^2}\,dx\,dy$

Kako naša data kriva pravi površinu $S: z=\sqrt{x^2+y^2}$ u prvom oktantu imamo

$$K=\iint\limits_S 3x(x^2+y^2)\,dx\,dy$$

Prisjetimo se kako se računa površinski integral II vrste npr. $I=\iint\limits_S R(x,y,z)\,dx\,dy$. Neka je $\vec{n}$ vektor normale na površ S, i neka neka je γ ugao koji $\vec{n}$ gradi sa z-osom, i neka je D projekcija površi S na xOy ravan. Tada

$$I=\iint\limits_S R(x,y,z)\,dx\,dy=\pm\iint\limits_D R(x,y,z(x,y))\,dx\,dy$$ gdje predznak ispred integrala zavisi od $\cos\gamma$ (za $\cos\gamma>0$ +, $\cos\gamma<0$ −).

Mi posmatramo vanjsku stranu površi, iz čega možemo zaključiti (sa slike) da je $\gamma\in(\frac{\pi}{2},\pi)$ pa je $\cos\gamma<0$.
Projekcija D površi S je data u sklopu zadatka (vidi sliku) ($\square A_1B_1C_1O$)

$$D:\begin{cases}0\le x\le 2\\ 0\le y\le 1\end{cases}\qquad K=\iint\limits_S 3x(x^2+y^2)\,dx\,dy=-\iint\limits_D 3x(x^2+y^2)\,dx\,dy=$$

$$=-3\int_0^1 dy\int_0^2 (x^3+xy^2)\,dx=-3\int_0^1\left(\frac{1}{4}x^4\Big|_0^2+\frac{1}{2}x^2y^2\Big|_0^2\right)dy=$$

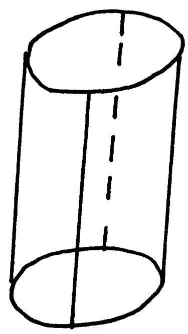
$$=-3\int_0^1(4+2y^2)\,dy=-3\left(4y\Big|_0^1+\frac{2}{3}y^3\Big|_0^1\right)=-12-2=-14$$ traženo rješenje

Uz pomoć formule Stoksa, izračunati krivolinijski integral

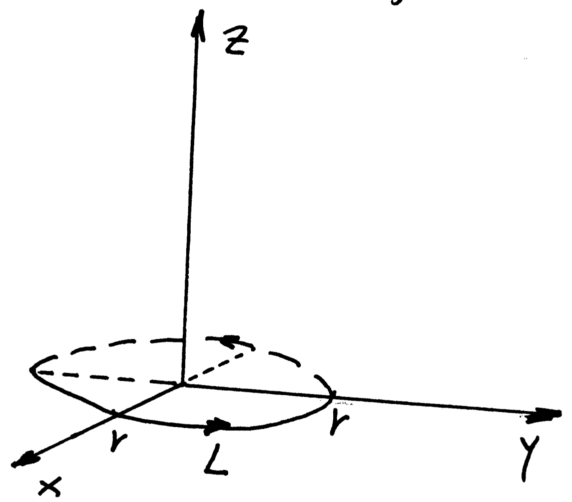
$$I = \oint_L x^2 y^3 \, dx + dy + z \, dz$$

gdje je L krug dat sa $x^2+y^2=r^2$ i $z=0$ $(r>0)$ (L je pozitivno orjentisana kriva ukoliko se posmatra sa pozitivnog dijela z-ose).

Rj: U xOy-ravni linija $x^2+y^2=r^2$ je krug sa centrom u tački $(0,0)$ poluprečnika r, a u prostoru to je cilindar



Krivu L u prostoru nije teško skicirati



Prisjetimo se formule Stoksa

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

Imamo

$$I = \oint_L x^2 y^3 \, dx + dy + z\,dz = \iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y^3 & 1 & z \end{vmatrix} =$$

$0-0$
$0-0$
$0-3x^2y^2$

$$= \iint_S (-3) x^2 y^2 \, dx\,dy = -3 \iint_D x^2 y^2 \, dx\,dy = $$

uvedimo polarne koordinate

$x = \rho \cos\varphi$
$y = \rho \sin\varphi$
$dx\,dy = \rho\, d\rho\, d\varphi$

$D \xrightarrow{\text{transf.}} D': \begin{cases} 0 \le \rho \le r \\ 0 \le \varphi \le 2\pi \end{cases}$

$$= -3 \iint_{D'} \rho^2\cos^2\varphi\, \rho^2\sin^2\varphi\, \rho\, d\rho\, d\varphi = -3\int_0^r \rho^5 d\rho \int_0^{2\pi} \frac{1}{4}\cdot \underbrace{4\cos^2\varphi\sin^2\varphi}_{(2\sin\varphi\cos\varphi)^2} d\varphi =$$

$$= -\frac{3}{4}\int_0^r \rho^5 d\rho \int_0^{2\pi} \sin^2 2\varphi\, d\varphi$$

$$\int_0^r \rho^5 d\rho = \frac{1}{6}\rho^6\Big|_0^r = \frac{1}{6}r^6$$

$$\int_0^{2\pi} \sin^2 2\varphi\, d\varphi = \left|\begin{array}{l} 1 = \cos^2 2\varphi + \sin^2 2\varphi \\ \cos 4\varphi = \cos^2 2\varphi - \sin^2 2\varphi \\ \hline 2\sin^2 2\varphi = 1 - \cos 4\varphi \end{array}\right| = \frac{1}{2}\int_0^{2\pi} (1 - \cos 4\varphi)\, d\varphi = \quad \ldots \quad = \pi$$

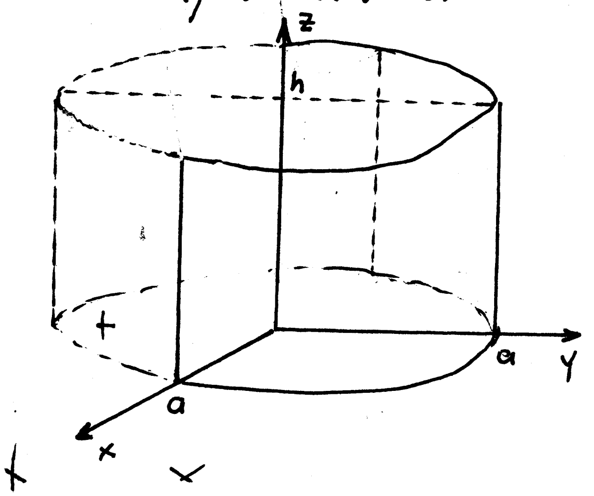
Prema tome

$$I = -\frac{3}{4}\cdot\frac{1}{6}r^6\cdot\pi = -\frac{1}{8}r^6\pi$$ traženo rješenje

(#) Uz pomoć formule Gauss-Ostrogradski izračunati površinski integral $I = \oint\!\!\oint_S 4x^3 dy\,dz + 4y^3 dx\,dz - 6z^4 dx\,dy$

gdje je S vanjska strana cilindra $x^2+y^2=a^2$ koji se nalazi između ravni $z=0$ i $z=h$.

Rj. Skicirajmo dati cilindar



Prisjetimo se formule Gauss-Ostrogradski

$$\oint\!\!\oint_S P(x,y,z)dy\,dz + Q(x,y,z)dx\,dz + R(x,y,z)dx\,dy = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

Ω - unutrašnjost oblasti S

$P(x,y,z) = 4x^3 \qquad \frac{\partial P}{\partial x} = 12x^2$

$Q(x,y,z) = 4y^3 \qquad \frac{\partial Q}{\partial y} = 12y^2$

$R(x,y,z) = 6z^4 \qquad \frac{\partial R}{\partial z} = 24z^3$

$$\oint\!\!\oint_S 4x^3 dy\,dz + 4y^3 dx\,dz - 6z^4 dx\,dy = 12\iiint_\Omega (x^2+y^2-2z^3)\,dx\,dy\,dz =$$

$$= \left| \begin{array}{ll} \text{uvedimo cilindrične koordinate} & \\ x = r\cos\varphi & \Omega \xrightarrow{\text{transformiše}} \Omega' \\ y = r\sin\varphi & 0 \le r \le a \\ z = z & 0 \le \varphi \le 2\pi \\ dx\,dy\,dz = r\,dr\,d\varphi\,dz & 0 \le z \le h \\ x^2+y^2 = \ldots = r^2 & \end{array} \right| = 12\iiint_{\Omega'} (r^2 - 2z^3)\,r\,dr\,d\varphi\,dz =$$

$$= 12\int_0^{2\pi} d\varphi \int_0^a dr \int_0^h (r^3 - 2rz^3)\,dz = 12\int_0^{2\pi} d\varphi \int_0^a \left(r^3 z\Big|_0^h - 2r\cdot\frac{1}{4}z^4\Big|_0^h\right) dr$$

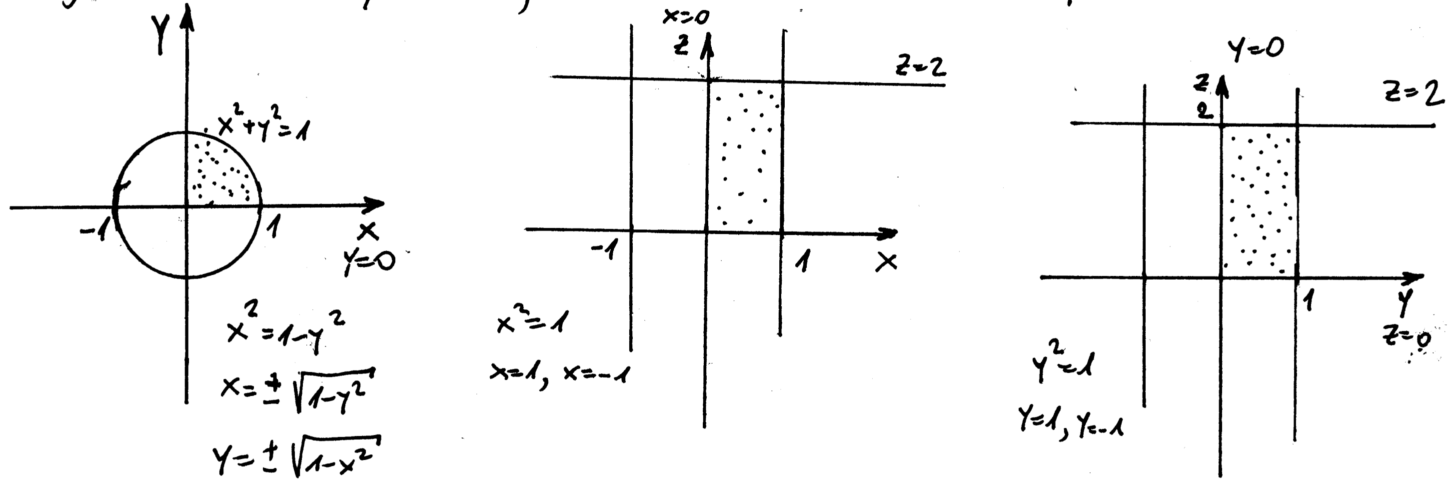
$$= 12\int_0^{2\pi} d\varphi \int_0^a \left(r^3 h - \frac{1}{2} r h^4\right) dr = 12\,\varphi\Big|_0^{2\pi} \left(h\frac{1}{4}r^4\Big|_0^a - \frac{1}{2}h^4\cdot\frac{1}{2}r^2\Big|_0^a\right) =$$

$$= 24\pi\cdot\frac{1}{4}h\,(a^4 - h^3 a^2) = 6\pi h a^2 (a^2 - h^3)$$ traženo rješenje

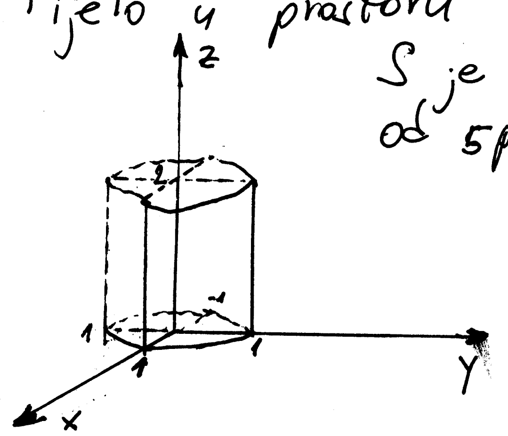
(#) Izračunati površinski integral $\iint\limits_S xz\,dy\,dz + xy\,dz\,dx + yz\,dx\,dy$,

ako je S vanjska strana ~~(omotača)~~ tijela koje pripada prvom oktantu i ograničeno je cilindrom $x^2+y^2=1$, te ravnima $x=0$, $y=0$, $z=0$, $z=2$.

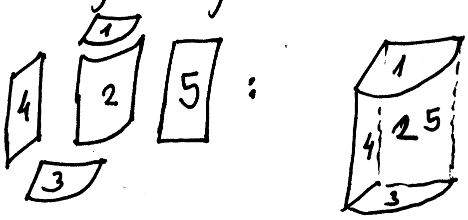
Rj. Skicirajmo dato tijelo. Odredimo prvo presjeke datog tijela sa xOy-ravni, sa xOz-ravni i sa yOz-ravni:



Tijelo u prostoru



S je vanjska strana tijela tj. S se sastoji od 5pet površina



Kako je S zatvorena površina to možemo upotrebiti formulu Gauss-Ostrogradski:

$$\iint\limits_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy = \iiint\limits_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

$P(x,y,z)=xz$, $\frac{\partial P}{\partial x}=z$, $Q(x,y,z)=xy$, $\frac{\partial Q}{\partial y}=x$, $R(x,y,z)=yz$, $\frac{\partial R}{\partial z}=y$

$$\iint\limits_S xz\,dy\,dz + xy\,dz\,dx + yz\,dx\,dy = \left|\begin{array}{l}\text{Formula}\\ \text{Gauss-}\\ \text{Ostrogradski}\end{array}\right| = \iiint\limits_\Omega (x+y+z)\,dx\,dy\,dz =$$

$$= \left|\begin{array}{l}\text{uvedimo cilindrične koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi \quad\quad \Omega \xrightarrow{\text{transformiše}} \Omega' : \begin{cases} 0\le r\le 1\\ 0\le\varphi\le\frac{\pi}{2}\\ 0\le z\le 2\end{cases}\\ z=z\\ dx\,dy\,dz = r\,dr\,d\varphi\,dz\end{array}\right| = \int_0^2 dz \int_0^{\pi/2} d\varphi \int_0^1 (r\cos\varphi + r\sin\varphi + z)\,r\,dr =$$

$$= \int_0^2 dz \int_0^{\pi/2} d\varphi \int_0^1 (r^2 \cos\varphi + r^2 \sin\varphi + r z)\, dr =$$

$$= \int_0^2 dz \int_0^{\pi/2} \left(\frac{1}{3} r^3 \Big|_0^1 \cos\varphi + \frac{1}{3} r^3 \Big|_0^1 \sin\varphi + \frac{1}{2} r^2 \Big|_0^1 z \right) d\varphi =$$

$$= \int_0^2 dz \int_0^{\pi/2} \left(\frac{1}{3} \cos\varphi + \frac{1}{3} \sin\varphi + \frac{1}{2} z \right) d\varphi =$$

$$= \int_0^2 \left(\frac{1}{3} \sin\varphi \Big|_0^{\frac{\pi}{2}} - \frac{1}{3} \cos\varphi \Big|_0^{\pi/2} + \frac{1}{2} z \varphi \Big|_0^{\frac{\pi}{2}} \right) dz$$

$$= \int_0^2 \left(\frac{1}{3} + \frac{1}{3} + \frac{\pi}{4} z \right) dz = \int_0^2 \left(\frac{2}{3} + \frac{\pi}{4} z \right) dz =$$

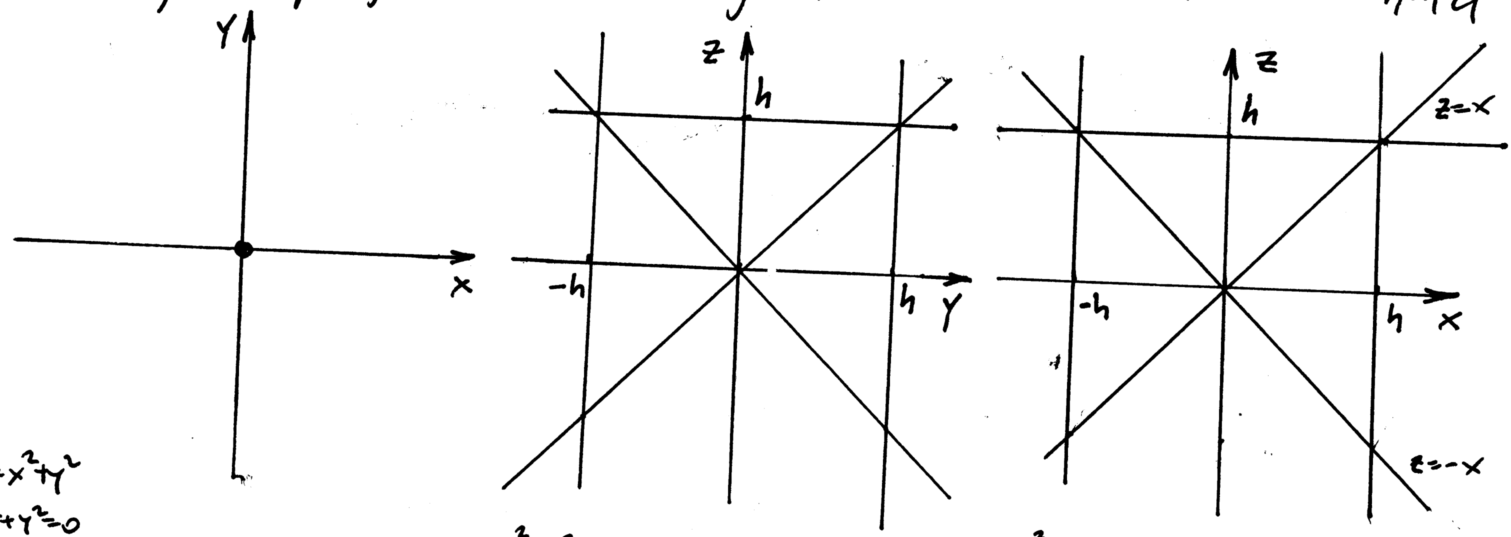
$$= \frac{2}{3} z \Big|_0^2 + \frac{\pi}{4} \cdot \frac{1}{2} z^2 \Big|_0^2 = \frac{4}{3} + \frac{\pi}{2}$$

traženo rješenje

(#) Izračunati površinski integral $\iint_{S^+} x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy$

gdje je S^+ spoljašnja strana kupe određene omotačem $z^2 = x^2+y^2$, $0 \le z \le h$ i osnovom $x^2+y^2 \le h^2$, $z=h$ za fiksirano $h>0$.

Rj. Skicirajmo presjeke datih figura sa koordinatnim ravnima



$z^2=x^2+y^2$
$x^2+y^2=0$

$x^2+y^2 \le h^2$
$z=h$

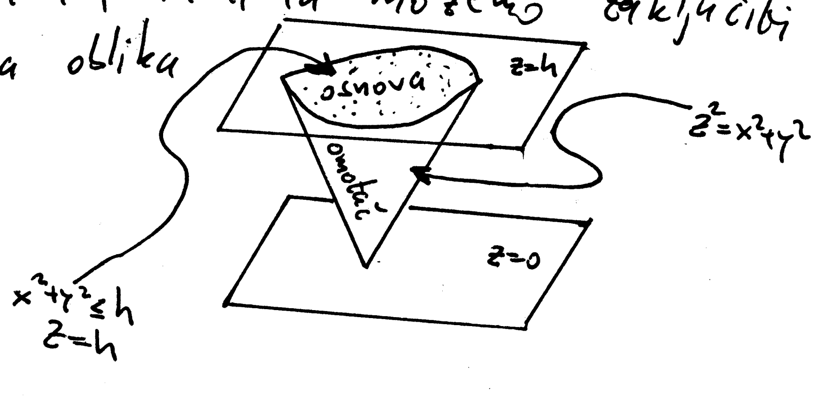
$z^2=x^2+y^2$
$y^2=z^2$
$z=\pm y$

$x^2+y^2 \le h^2$
$z=h$
$y^2 \le h$
$z=h$

$z^2=x^2+y^2$
$z^2=x^2$
$z=\pm x$

$x^2+y^2 \le h^2$
$z=h$
$x^2 \le h^2$
$z=h$

Sa presjeka figura sa koordinatnim ravnima možemo zaključiti da je $z^2=x^2+y^2$, $0\le z\le h$ kupa oblika dok je osnova u stvari "poklopac" kupe (vidi sliku)



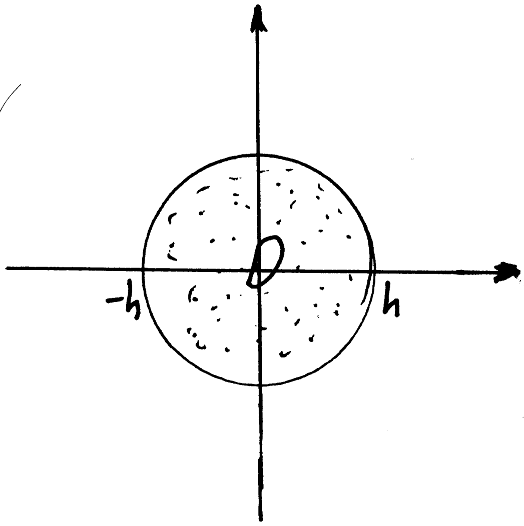
Kako imamo zatvorenu figuru možemo upotrebiti formulu Gauss-Ostrogradskog.

$$\iint_{S+} x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy \overset{\text{form. Gauss-Ostr.}}{=}$$

$$= \iiint_{\Omega} (2x+2y+2z)\,dx\,dy\,dz = 2\iiint_{\Omega} (x+y+z)\,dx\,dy\,dz$$

gdje je Ω unutrašnjost kupe koja je ograničena sa ravnima $z=0$ i $z=h$ (vidi sliku).

Ortogonalna projekcija oblasti Ω na xOy-ravan je krug sa centrom u koordinatnom početku poluprečnika h.



$$I = 2\iiint_{\Omega} (x+y+z)\,dx\,dy\,dz = 2\iint_{D} dx\,dy \int_{\sqrt{x^2+y^2}}^{h} (x+y+z)\,dz$$

$$= 2\iint_{D} \left(xz\Big|_{\sqrt{x^2+y^2}}^{h} + yz\Big|_{\sqrt{x^2+y^2}}^{h} + \frac{1}{2}z^2\Big|_{\sqrt{x^2+y^2}}^{h} \right) dx\,dy =$$

$$= 2\iint_{D} \left(hx + hy + \frac{1}{2}h^2 - x\sqrt{x^2+y^2} - y\sqrt{x^2+y^2} - \frac{1}{2}(x^2+y^2) \right) dx\,dy =$$

uvedimo polarne koordinate: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $dx\,dy = \rho\,d\rho\,d\varphi$, $x^2+y^2 = \ldots = \rho^2$, $D \xrightarrow{\text{transf.}} D'$

$$= 2\iint_{D'} \left(h\rho\cos\varphi + h\rho\sin\varphi + \frac{1}{2}h^2 - \rho\cos\varphi\cdot\rho - \rho\sin\varphi\cdot\rho - \frac{1}{2}\rho^2 \right)\rho\,d\rho\,d\varphi$$

$$= 2\int_0^{2\pi} d\varphi \int_0^{h} \left(h\rho^2\cos\varphi + h\rho^2\sin\varphi + \frac{1}{2}h^2\rho - \rho^3\cos\varphi - \rho^3\sin\varphi - \frac{1}{2}\rho^3 \right) d\rho$$

$$= \underset{\ldots}{\text{za vježbu}} = \frac{h^4}{12}\int_0^{2\pi} (2\cos\varphi + 2\sin\varphi + 3)\,d\varphi = \ldots = \frac{1}{2}h^4\pi$$ traženo rješenje

(#) Prvo izračunati integral $\int_0^\infty e^{-x}\sin(\alpha x)\,dx$ pa poslije toga dobijeni rezultat iskoristiti i koristeći metodu diferenciranja po parametru izračunati

$$G(\alpha)=\int_0^\infty x e^{-x}\cos(\alpha x)\,dx .$$

Rj.

$$I(\alpha)=\int_0^\infty e^{-x}\sin(\alpha x)\,dx = \left| \begin{matrix} u=e^{-x} & dv=\sin\alpha x\,dx \\ du=-e^{-x}dx & v=-\frac{1}{\alpha}\cos\alpha x \end{matrix} \right| =$$

$$= -\frac{1}{\alpha}e^{-x}\cos\alpha x\Big|_0^\infty - \frac{1}{\alpha}\int_0^\infty e^{-x}\cos(\alpha x)\,dx = \left| \begin{matrix} u=e^{-x} & dv=\cos\alpha x\,dx \\ du=-e^{-x} & v=\frac{1}{\alpha}\sin\alpha x \end{matrix} \right|$$

ovdje podrazumijevano da se računa $\lim_{A\to\infty}(e^{-x}\cos\alpha x)\Big|_0^A$

$$=\left(0+\frac{1}{\alpha}\cdot 1\right) - \underbrace{\frac{1}{\alpha^2}e^{-x}\sin\alpha x\Big|_0^\infty}_{0-0} - \frac{1}{\alpha^2}\int_0^\infty e^{-x}\sin\alpha x\,dx$$

ovdje podrazumijevam da se računa $\lim_{A\to\infty}(e^{-x}\sin\alpha x)\Big|_0^A$

$$\Rightarrow \quad I(\alpha)=\frac{1}{\alpha}-\frac{1}{\alpha^2}I(\alpha)$$

$$I(\alpha)+\frac{1}{\alpha^2}I(\alpha)=\frac{1}{\alpha} \quad \Rightarrow \quad \left(1+\frac{1}{\alpha^2}\right)I(\alpha)=\frac{1}{\alpha}$$

$$\frac{\alpha^2+1}{\alpha^2}I(\alpha)=\frac{1}{\alpha} \quad /\cdot\alpha$$

$$I(\alpha)=\frac{\alpha}{\alpha^2+1}$$

Označimo sa $F(\alpha)=\int_0^\infty e^{-x}\sin(\alpha x)\,dx=\frac{\alpha}{\alpha^2+1}$.

Kako je $\left(e^{-x}\sin(\alpha x)\right)'_\alpha = x e^{-x}\cos\alpha x$ i

$$\left(\frac{\alpha}{\alpha^2+1}\right)'_\alpha = \frac{1\cdot(\alpha^2+1)-\alpha\cdot 2\alpha}{(\alpha^2+1)^2} = \frac{1-\alpha^2}{(\alpha^2+1)^2}$$

To je

$$F'(\alpha) = \int_0^\infty x e^{-x} \cos\alpha x \, dx = \frac{1-\alpha^2}{(\alpha^2+1)^2}$$

Pa je

$$G(\alpha) = \int_0^\infty x e^{-x} \cos\alpha x \, dx = \frac{1-\alpha^2}{(\alpha^2+1)^2}$$

traženo rješenje

(#) Date su vrijednosti dva integrala $(\alpha>0)$

$$\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx=\frac{\pi}{2}e^{-\alpha} \quad \text{i} \quad \int_0^\infty \frac{\sin\alpha x}{x}\,dx=\frac{\pi}{2}.$$

Koristeći date jednakosti uz pomoć metode diferenciranja po parametru izračunati

$$\int_0^\infty \frac{\sin\alpha x}{x(1+x^2)}\,dx.$$

Rj: Označimo sa $F(\alpha)=\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx=\frac{\pi}{2}e^{-\alpha}$.

Kako je $\left(\frac{\cos\alpha x}{1+x^2}\right)'_\alpha=\frac{-x\sin\alpha x}{1+x^2}$ i $\left(\frac{\pi}{2}e^{-\alpha}\right)'_\alpha=-\frac{\pi}{2}e^{-\alpha}$

To je $F'_\alpha=\int_0^\infty \frac{-x\sin\alpha x}{1+x^2}\,dx=-\frac{\pi}{2}e^{-\alpha}$ tj. $\int_0^\infty \frac{x\sin\alpha x}{1+x^2}\,dx=\frac{\pi}{2}e^{-\alpha}$

Sad primjetimo da je

$$\frac{x\sin\alpha x}{1+x^2}=\frac{x^2\sin\alpha x \overset{+\sin\alpha x-\sin\alpha x}{}}{x(1+x^2)}=\frac{x^2\sin\alpha x+\sin\alpha x}{x(1+x^2)}-\frac{\sin\alpha x}{x(1+x^2)}=$$

$$=\frac{\sin\alpha x\cdot(x^2+1)}{x\cdot(1+x^2)}-\frac{\sin\alpha x}{x(1+x^2)}=\frac{\sin\alpha x}{x}-\frac{\sin\alpha x}{x(1+x^2)}$$

Pa imamo

$$\int_0^\infty \frac{\sin\alpha x}{x(1+x^2)}\,dx=\overbrace{\int_0^\infty \frac{\sin\alpha x}{x}\,dx}^{\frac{\pi}{2}}-\overbrace{\int_0^\infty \frac{x\sin\alpha x}{1+x^2}\,dx}^{\frac{\pi}{2}e^{-\alpha}}=\frac{\pi}{2}\left(1-e^{-\alpha}\right)$$

Metodom diferenciranja po parametru izračunati integral

$$\int_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}}\,dx \quad (a^2<1).$$

Rj: Prisjetimo se

Ako je dat integral $F(\alpha)=\int_a^b f(x,\alpha)\,dx$ u kome granice a i b ne zavise od parametra α i u kome f ima neprekidan parcijalni izvod po α tada

$$F'(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx.$$

U našem slučaju f-ja f je $f=\dfrac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}}$

$$\frac{\partial f}{\partial a}=\frac{1}{x^2\sqrt{1-x^2}}\cdot\frac{-2ax^2}{1-a^2x^2}=\frac{-2a}{(1-a^2x^2)\sqrt{1-x^2}}$$

$$F'(a)=\int_0^1\frac{-2a}{(1-a^2x^2)\sqrt{1-x^2}}\,dx=\left|\begin{array}{l} x=\sin t \\ dx=\cos t\,dt \quad x\Big|_0^1 \Rightarrow t\Big|_0^{\pi/2} \\ 1-x^2=1-\sin^2 t=\cos^2 t\end{array}\right|=$$

$$=(-2a)\int_0^{\pi/2}\frac{\cos t\,dt}{(1-a^2\sin^2 t)\cos t}=\left|\begin{array}{l} \operatorname{tg} t=z \qquad \sin^2 t=\frac{\sin^2 t\ /:\cos^2 t}{\sin^2 t+\cos^2 t\ /:\cos^2 t} \\ t=\operatorname{arctg} z \qquad \sin^2 t=\frac{z^2}{1+z^2} \\ dt=\frac{dz}{1+z^2} \qquad t\Big|_0^{\pi/2}\Rightarrow z\Big|_0^{\infty}\end{array}\right|$$

$$=(-2a)\int_0^\infty\frac{\frac{dz}{1+z^2}}{1-\frac{a^2z^2}{1+z^2}}=(-2a)\int_0^\infty\frac{\frac{dz}{1+z^2}}{\frac{1+z^2-a^2z^2}{1+z^2}}=(-2a)\int_0^\infty\frac{dz}{(1-a^2)z^2+1}=$$

$$= \frac{-2a}{1-a^2} \int_0^{\infty} \frac{dz}{z^2 + \frac{1}{1-a^2}} = \frac{-2a}{1-a^2} \cdot \frac{1}{\sqrt{\frac{1}{1-a^2}}} \operatorname{arctg} \frac{z}{\frac{1}{\sqrt{1-a^2}}} \Bigg|_0^{\infty}$$

$$= \frac{-2a\sqrt{1-a^2}}{1-a^2} \cdot \frac{\pi}{2} = \frac{-a\pi}{\sqrt{1-a^2}}$$

tj. $F'_a = -\pi \dfrac{a}{\sqrt{1-a^2}}$

$$F(a) = \int F'_a \, da = -\pi \int \frac{a}{\sqrt{1-a^2}} \, da = \left| \begin{matrix} d(1-a^2) = -2a\,da \\ -a\,da = \frac{1}{2} d(a^2-1) \end{matrix} \right|$$

$$= \frac{\pi}{2} \int (1-a^2)^{-\frac{1}{2}} d(1-a^2) = \frac{\pi}{2} \cdot \frac{(1-a^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = \pi\sqrt{1-a^2} + c$$

$$\left. \begin{matrix} F(a) = \pi\sqrt{1-a^2} + c \\ F(a) = \displaystyle\int_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}} \, dx \end{matrix} \right\} \Rightarrow \left. \begin{matrix} F(0) = \pi\sqrt{1-0} + c \\ F(0) = 0 \end{matrix} \right\} \Rightarrow$$

$$\Rightarrow \quad \pi + c = 0 \quad \Rightarrow \quad c = -\pi$$

Prema tome

$$\int_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}} \, dx = \pi\left(\sqrt{1-a^2} - 1\right)$$

traženo rješenje

(#) Metodom diferenciranja po parametru izračunati integral $\int_0^1 \frac{\operatorname{arctg} ax}{x\sqrt{1-x^2}}\,dx$.

Rj. Prisjetimo se:

Ako je dat integral $F(\alpha) = \int_a^b f(x,\alpha)\,dx$ u kome granice a i b ne zavise od parametra α tada $F'(\alpha) = \int_a^b f'_\alpha(x,\alpha)\,dx$.

U našem slučaju f-ja f je $f(x,a) = \frac{\operatorname{arctg} ax}{x\sqrt{1-x^2}}$

$$f'_a = \frac{1}{x\sqrt{1-x^2}} \cdot \frac{x}{1+(ax)^2} = \frac{1}{(1+a^2x^2)\sqrt{1-x^2}}$$

$$F(a) = \int_0^1 \frac{\operatorname{arctg} ax}{x\sqrt{1-x^2}}\,dx$$

$$F'_a = \int_0^1 \frac{dx}{(1+a^2x^2)\sqrt{1-x^2}} = \left| \begin{array}{ll} x = \sin t & x\big|_0^1 \Rightarrow t\big|_0^{\frac{\pi}{2}} \\ dx = \cos t\,dt & \\ \sqrt{1-x^2} = \sqrt{\cos^2 t} = |\cos t| & \end{array} \right|$$

$$= \int_0^{\pi/2} \frac{\cos t\,dt}{(1+a^2\sin^2 t)\cos t} = \int_0^{\pi/2} \frac{dt}{1+a^2\sin^2 t} = \left| \begin{array}{ll} \operatorname{tg} t = z & \sin^2 t = \frac{z^2}{1+z^2} \\ t = \operatorname{arctg} z & \\ dt = \frac{dz}{1+z^2} & t\big|_0^{\pi/2} \Rightarrow z\big|_0^{\infty} \end{array} \right|$$

$$= \int_0^{\infty} \frac{\frac{dz}{1+z^2}}{1+a^2\frac{z^2}{1+z^2}} = \int_0^{\infty} \frac{\frac{dz}{1+z^2}}{\frac{1+z^2+a^2z^2}{1+z^2}} = \int_0^{\infty} \frac{dz}{(a^2+1)z^2+1} =$$

$$= \frac{1}{a^2+1} \int_0^\infty \frac{dz}{z^2 + \frac{1}{a^2+1}} = \frac{1}{a^2+1} \cdot \frac{1}{\sqrt{\frac{1}{a^2+1}}} \operatorname{arc\,tg} \frac{z}{\sqrt{\frac{1}{a^2+1}}} \Bigg|_0^\infty$$

$$= \frac{\sqrt{a^2+1}}{a^2+1} \cdot \frac{\pi}{2} = \frac{\frac{\pi}{2}}{\sqrt{a^2+1}} \quad \Rightarrow \quad F'_a = \frac{\pi/2}{\sqrt{a^2+1}}$$

$$F(a) = \frac{\pi}{2} \int \frac{da}{\sqrt{a^2+1}} = \frac{\pi}{2} \ln \left| a + \sqrt{a^2+1} \right| + C$$

Prema tome

$$\left.\begin{aligned} F(a) &= \frac{\pi}{2} \ln \left| a + \sqrt{a^2+1} \right| + C \\ F(a) &= \int_0^1 \frac{\operatorname{arc\,tg} ax}{x\sqrt{1-x^2}}\, dx \end{aligned}\right\} \Rightarrow \quad \begin{aligned} F(0) &= C \\ F(0) &= 0 \end{aligned} \quad C = 0$$

Prema tome

$$\int_0^1 \frac{\operatorname{arc\,tg} ax}{x\sqrt{1-x^2}} = \frac{\pi}{2} \ln \left| a + \sqrt{a^2+1} \right|$$

(#) Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v} = 2x(y^2+z^2)\vec{i} + 2y(x^2+z^2)\vec{j} + 2z(x^2+y^2)\vec{k}.$$

Rj. Vektorsko polje $\vec{v}$ je potencijalno ako je $rot\,\vec{v} = \vec{0}$.
Rotor vektorskog polja $rot\,\vec{v}$ se računa ..

$$rot\,\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

(vektorski proizvod Nabla (∇) operatora i vektorskog polja $\vec{v}$)

$V_x = 2x(y^2+z^2)$ $\quad V_y = 2y(x^2+z^2)$ $\quad V_z = 2z(x^2+y^2)$

$\frac{\partial V_x}{\partial y} = 4xy$ $\quad \frac{\partial V_y}{\partial x} = 4xy$ $\quad \frac{\partial V_z}{\partial x} = 4xz$

$\frac{\partial V_x}{\partial z} = 4xz$ $\quad \frac{\partial V_y}{\partial z} = 4yz$ $\quad \frac{\partial V_z}{\partial y} = 4yz$

$$rot\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \vec{i}(4yz-4yz) - \vec{j}(4xz-4xz) + \vec{k}(4xy-4xy)$$

$= (0,0,0) = \vec{0}$ vektorsko polje je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\vec{v} = grad\,u$.

$$grad\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x(y^2+z^2) \\ \frac{\partial u}{\partial y} &= 2y(x^2+z^2) \\ \frac{\partial u}{\partial z} &= 2z(x^2+y^2) \end{aligned}\right\} \ldots (1)$$

$u = u(x,y,z)$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$u = \int 2x(y^2+z^2)\,dx + \varphi(y,z)$

$u = x^2(y^2+z^2) + \varphi(y,z)$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= 2yx^2 + \varphi'_y \\ \frac{\partial u}{\partial z} &= 2x^2z + \varphi'_z \end{aligned}\right\} \ldots (2)$$

(1) i (2) $\Rightarrow$ $\varphi'_y = 2yz^2$, $\varphi'_z = 2zy^2 \ldots (*)$

Odredimo f-ju φ: $\varphi = \int 2yz^2\,dy + \psi(z)$

$\varphi = y^2z^2 + \psi(z)$

$\varphi' = 2y^2z + \psi' \ldots (**)$

$(*)$ i $(**)$ $\Rightarrow$ $\psi'_z = 0 \Rightarrow \psi(z) = C$

$\Rightarrow \varphi = y^2z^2 + C \Rightarrow u = x^2y^2 + x^2z^2 + y^2z^2 + C$

Potencijal vektorskog polja je $u = x^2y^2 + x^2z^2 + y^2z^2 + C$

(#) Odrediti brojeve a i b tako da vektorsko polje $\vec{v} = (yz + axy, xz + bx^2 + yz^2, axy + y^2z)$ bude potencijalno i za dobijeno polje izračunati njegovu cirkulaciju duž pravoliniske konture od tačke $A(1,1,1)$ prema tački $B(2,2,2)$.

Rj: Za vektorsko polje $\vec{v}$ ($\vec{v} = (v_x, v_y, v_z)$) kažemo da je potencijalno ako je $\operatorname{rot}\vec{v} = \vec{0}$. Znamo da

$$\operatorname{rot}\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz + axy & xz + bx^2 + yz^2 & axy + y^2z \end{vmatrix} =$$

$$= (ax + 2yz - x - 2yz, -(ay - y), z + 2bx - z - ax)$$

$$= (ax - x, y - ay, 2bx - ax).$$

$$\operatorname{rot}\vec{v} = \vec{0} \Rightarrow \begin{array}{l} ax - x = 0 \\ y - ay = 0 \\ 2bx - ax = 0 \end{array} \qquad \begin{array}{l} a = 1 \\ b = \frac{1}{2} \end{array}$$

Za $a = 1$ i $b = \frac{1}{2}$ vektorsko polje $\vec{v}$ je potencijalno polje.

Cirkulaciju vektorskog polja $\vec{v}$ duž krive c tražimo po formuli

$$C = \int_c \vec{v}\, d\vec{r} = \int_c v_x dx + v_y dy + v_z dz$$

Kriva c je dio prave od tačke $A(1,1,1)$ (x_1, y_1, z_1) do tačke $B(2,2,2)$ (x_2, y_2, z_2).

Kako glasi jednačina prave u prostoru kroz dvije tačke?

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \Longrightarrow \frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1} \quad (= t) \qquad \begin{array}{l} x - 1 = t \\ y - 1 = t \\ z - 1 = t \end{array}$$

Krivulja c u parametarskom obliku

$$c: \begin{cases} x = t+1 \\ y = t+1 \\ z = t+1 \\ 0 \le t \le 1 \end{cases} \qquad \begin{matrix} dx = dt \\ dy = dt \\ dz = dt \end{matrix}$$

U našem slučaju

$$C = \int_c (yz + xy)\,dx + (xz + \tfrac{1}{2}x^2 + yz^2)\,dy + (xy + y^2 z)\,dz =$$

$$= \int_0^1 \Big[(t+1)^2 + (t+1)^2 + (t+1)^2 + \tfrac{1}{2}(t+1)^2 + (t+1)^3 + (t+1)^2 + (t+1)^3\Big]\,dt =$$

$$= \Big| \, d(t+1) = dt \, \Big| = \int_0^1 \Big[\tfrac{9}{2}(t+1)^2 + 2(t+1)^3\Big]\,d(t+1) =$$

$$= \frac{9}{2}\,\frac{(t+1)^3}{3}\Big|_0^1 + 2\cdot\frac{(t+1)^4}{4}\Big|_0^1 = \frac{9}{6}(8-1) + \frac{1}{2}(16-1)$$

$$= \frac{63}{6} + \frac{15\cdot 3}{2\cdot 3} = \frac{108}{6} = 18$$ traženo rješenje

(#) Neka f-je $g, h : \mathbb{R}^3 \to \mathbb{R}$ ispunjavaju

$$\Delta g(x,y,z) = 0 \quad i \quad \Delta h(x,y,z) = 0$$

gdje je $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ LAPLACE-ov operator.

Za f-ju $f : \mathbb{R}^3 \to \mathbb{R}$ datu sa

$$f(x,y,z) = g(x,y,z) + (x^2+y^2+z^2)\, h(x,y,z)$$

izračunati $\Delta \Delta f(x,y,z)$.

Rj:

$$\Delta f(x,y,z) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$f(x,y,z) = g(x,y,z) + (x^2+y^2+z^2)\, h(x,y,z)$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} + 2x\, h + (x^2+y^2+z^2)\frac{\partial h}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial x^2} + 2h + 2x\frac{\partial h}{\partial x} + 2x\frac{\partial h}{\partial x} + (x^2+y^2+z^2)\frac{\partial^2 h}{\partial x^2}$$

$$= \frac{\partial^2 g}{\partial x^2} + 2h + 4x\frac{\partial h}{\partial x} + (x^2+y^2+z^2)\frac{\partial^2 h}{\partial x^2} \qquad \dots(1)$$

slično bi dobili

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial x^2} + 2h + 4y\frac{\partial h}{\partial y} + (x^2+y^2+z^2)\frac{\partial^2 h}{\partial y^2} \qquad \dots(2) \quad i$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 g}{\partial z^2} + 2h + 4z\frac{\partial h}{\partial z} + (x^2+y^2+z^2)\frac{\partial^2 h}{\partial z^2} \qquad \dots(3)$$

Kada saberemo (1), (2) i (3) dobićemo

$$\Delta f = \underbrace{\Delta g}_{=0} + 6h + 4x\frac{\partial h}{\partial x} + 4y\frac{\partial h}{\partial y} + 4z\frac{\partial h}{\partial z} + (x^2+y^2+z^2)\underbrace{\Delta h}_{=0} =$$

$$= 6h(x,y,z) + 4x\frac{\partial}{\partial x}h(x,y,z) + 4y\frac{\partial}{\partial y}h(x,y,z) + 4z\frac{\partial}{\partial z}h(x,y,z)$$

$$\Delta f = 6h + 4x\frac{\partial h}{\partial x} + 4y\frac{\partial h}{\partial y} + 4z\frac{\partial h}{\partial z}$$

$$\Delta\Delta f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Delta f = \frac{\partial^2 \Delta f}{\partial x^2} + \frac{\partial^2 \Delta f}{\partial y^2} + \frac{\partial^2 \Delta f}{\partial z^2}$$

$$\frac{\partial \Delta f}{\partial x} = 6\frac{\partial h}{\partial x} + 4\frac{\partial h}{\partial x} + 4x\frac{\partial^2 h}{\partial x^2} + 4y\frac{\partial^2 h}{\partial x \partial y} + 4z\frac{\partial^2 h}{\partial x \partial z}$$

$$= 10\frac{\partial h}{\partial x} + 4x\frac{\partial^2 h}{\partial x^2} + 4y\frac{\partial^2 h}{\partial x \partial y} + 4z\frac{\partial^2 h}{\partial x \partial z}$$

$$\frac{\partial^2 \Delta f}{\partial x^2} = 10\frac{\partial^2 h}{\partial x^2} + 4\frac{\partial^2 h}{\partial x^2} + 4x\frac{\partial^3 h}{\partial x^3} + 4y\frac{\partial^2 h}{\partial x^2 \partial y} + 4z\frac{\partial^2 h}{\partial x^2 \partial z}$$

$$= 14\frac{\partial^2 h}{\partial x^2} + 4x\frac{\partial^3 h}{\partial x^3} + 4y\frac{\partial^3 h}{\partial x^2 \partial y} + 4z\frac{\partial^3 h}{\partial x^2 \partial z}$$

Slično

$$\frac{\partial^2 \Delta f}{\partial y^2} = 14\frac{\partial^2 h}{\partial y^2} + 4x\frac{\partial^3 h}{\partial x \partial y^2} + 4y\frac{\partial^3 h}{\partial y^3} + 4z\frac{\partial^3 h}{\partial y^2 \partial z}$$

$$\frac{\partial^2 \Delta f}{\partial z^2} = 14\frac{\partial^2 h}{\partial z^2} + 4x\frac{\partial^3 h}{\partial x \partial z^2} + 4y\frac{\partial^3 h}{\partial y \partial z^2} + 4z\frac{\partial^3 h}{\partial z^3}$$

Prema tome

$$\Delta\Delta f = 14\Delta h + 4x\frac{\partial}{\partial x}\Delta h + 4y\frac{\partial}{\partial y}\Delta h + 4z\frac{\partial}{\partial z}\Delta h$$

Kako je $\Delta h = 0$ prema postavci zadatka to je

$$\Delta\Delta f = 0$$

(#) Pokazati da je vektorsko polje

$$\vec{v} = (2x+y+z,\ x+2y+z,\ x+y+2z)$$

potencijalno i naći njegov potencijal.

Rj:

Ako je $\text{rot}\,\vec{v} = \vec{0}$ tada za polje $\vec{v}$ kažemo da je potencijalno polje.

$$\text{rot}\,\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+y+z & x+2y+z & x+y+2z \end{vmatrix}$$

$$= (1-1,\ -(1-1),\ 1-1) = (0,0,0)$$

vektorsko polje $\vec{v}$ je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\text{grad}\,u = \vec{v}$.

$$\text{grad}\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = (2x+y+z,\ x+2y+z,\ x+y+2z)$$

$$\frac{\partial u}{\partial x} = 2x+y+z \quad \dots(1)$$

$$\frac{\partial u}{\partial y} = x+2y+z \quad \dots(2)$$

$$\frac{\partial u}{\partial z} = x+y+2z \quad \dots(3)$$

$$(1) \Rightarrow \quad u = x^2 + xy + xz + \varphi(y,z)$$

$$\frac{\partial u}{\partial y} = x + \varphi'_y \quad \dots(4)$$

$$\frac{\partial u}{\partial z} = x + \varphi'_z \quad \dots(5)$$

$$(2) \text{ i } (4) \Rightarrow \quad \varphi'_y = 2y+z \Rightarrow \frac{\partial \varphi}{\partial y} = 2y+z \quad \dots(6)$$

$$(3) \text{ i } (5) \Rightarrow \quad \varphi'_z = y+2z \Rightarrow \frac{\partial \varphi}{\partial z} = y+2z \quad \dots(7)$$

$$(6) \Rightarrow \quad \varphi = y^2 + yz + \theta(z)$$

$$\frac{\partial \varphi}{\partial z} = y + \theta'_z \quad \dots(8)$$

$$(7) \text{ i } (8) \Rightarrow \quad \theta'_z = 2z$$

$$\theta = z^2 + C$$

Sad imamo $u = x^2+xy+xz+\varphi = x^2+xy+xz+y^2+yz+\theta = x^2+y^2+z^2+xy+zx+yz+C$

Prema tome traženi potencijal je $u = x^2+y^2+z^2+xy+zx+yz+C$

Dokazati da je vektorsko polje

$$\vec{v} = (z\cos zx - y\sin x,\ \cos x,\ x\cos zx)$$

potencijalno i izračunati cirkulaciju tog polja duž prave od tačke $O(0,0,0)$ do tačke $A(1,2,\pi)$.

Rj.

Ako je $\operatorname{rot}\vec{v} = \vec{0}$ tada kažemo da je $\vec{v}$ potencijalno polje.

$$\operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos zx - y\sin x & \cos x & x\cos zx \end{vmatrix}$$

$$= \left(0-0,\ -(\cos zx - zx\cos zx - \cos zx + xz\cos zx),\ -\sin x + \sin x\right)$$

$$= (0,0,0) \Rightarrow \vec{v} \text{ je potencijalno polje}$$

$$C = \int_C \vec{v}\, d\vec{r} = \int_C V_x dx + V_y dy + V_z dz \quad \text{cirkulacija vektorskog polja}$$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke

$O(0,0,0)$, $A(1,2,\pi)$ $\quad \frac{x}{1} = \frac{y}{2} = \frac{z}{\pi} \ (=t)$

$$C = \int_{\overline{OA}} (z\cos zx - y\sin x)dx + \cos x\,dy + x\cos zx\,dz = \left| \overline{OA}: \begin{cases} x = t \\ y = 2t \\ z = \pi t \\ 0 \le t \le 1 \end{cases} \quad dx = dt,\ dy = 2dt,\ dz = \pi dt \right|$$

$$= \int_0^1 \left[\pi t\cos\pi t^2 - 2t\sin t + 2\cos t + \pi t\cos\pi t^2\right] dt$$

$$= \int_0^1 (2\cos t - 2t\sin t + 2\pi t\cos\pi t^2)\, dt = \ldots = 2\sin(1) + 2\cos(1) - 2\sin(1) + 0 = 2\cos 1$$

\# Izračunati cirkulaciju vektorskog polja $\vec{v}=(1, xy^2, yz^2)$ duž konture $x^2+2y^2=4$, $z=2x$.

R.: Cirkulacija vektorskog polja $\vec{v}$ $(\vec{v}=(v_x, v_y, v_z))$ duž krive c je integral

$$C=\int_c \vec{v}\, d\vec{r} = \int_c v_x\, dx + v_y\, dy + v_z\, dz$$

Pokušajmo skicirati konturu c.

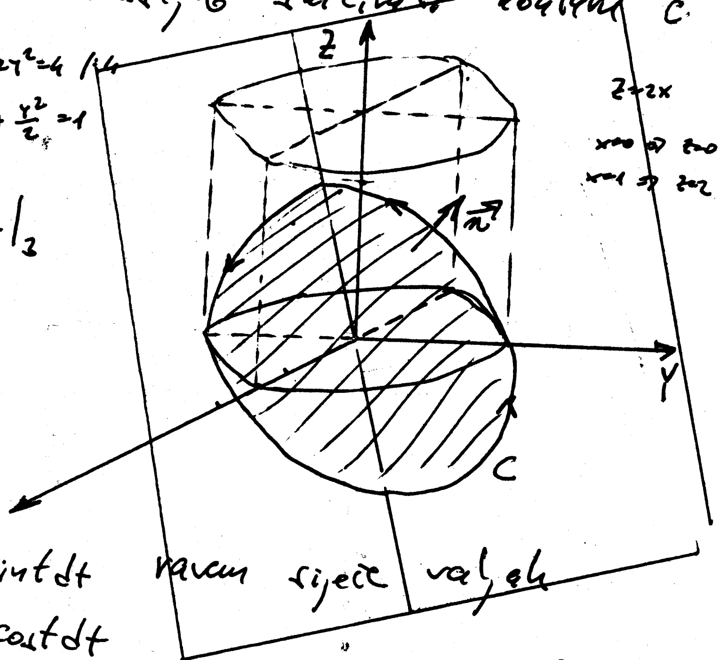
$x^2+2y^2=4 \;/:4$

$\frac{x^2}{4}+\frac{y^2}{2}=1$

$z=2x$

$x=0 \Rightarrow z=0$

$x=1 \Rightarrow z=2$



U našem slučaju

$$C=\int_c dx + xy^2\, dy + yz^2\, dz = I_1+I_2+I_3$$

parametriziramo konturu c

kako je $\left(\frac{x}{2}\right)^2+\left(\frac{y}{\sqrt{2}}\right)^2=1$ uvedimo smjene

$$\left.\begin{array}{l} \frac{x}{2}=\cos t \\ \frac{y}{\sqrt{2}}=\sin t \\ z=z \end{array}\right\} \Rightarrow \begin{array}{ll} x=2\cos t & dx=-2\sin t\, dt \\ y=\sqrt{2}\sin t & dy=\sqrt{2}\cos t\, dt \\ z=4\cos t & dz=-4\sin t\, dt \\ 0\le t\le 2\pi & \end{array}$$

$\sin^2 x+\cos^2 x=1$

$\cos^2 x-\sin^2 x=\cos 2x$

$2\sin^2 x=1-\cos 2x$

$2\cos^2 x=1+\cos 2x$

$$C=\int_0^{2\pi}\left(-2\sin t + 2\cos t\cdot 2\sin^2 t\cdot\sqrt{2}\cos t+\sqrt{2}\sin t\cdot 16\cdot\cos^2 t(-4\sin t)\right)dt$$

pojednostavimo računanje ovog integrala

$$I_1=\int_c dx=\int_0^{2\pi}-2\sin t\, dt=2\cos t\Big|_0^{2\pi}=2(1-1)=0$$

$4\cos^2 t\sin^2 t=(2\cos t\sin t)^2$

$$I_2=\int_c xy^2\, dy=\int_0^{2\pi}2\cos t\cdot 2\sin^2 t\cdot\sqrt{2}\cos t\, dt=4\sqrt{2}\int_0^{2\pi}\cos^2 t\sin^2 t\, dt=\sqrt{2}\int_0^{2\pi}\sin^2 2t\, dt$$

$$=\frac{\sqrt{2}}{2}\int_0^{2\pi}(1-\cos 2t)\, dt=\frac{\sqrt{2}}{2}\left(t\Big|_0^{2\pi}-\frac{1}{2}\sin t\Big|_0^{2\pi}\right)=\frac{\sqrt{2}}{2}(2\pi-0)=\pi\sqrt{2}$$

$$I_3=\int_c yz^2\, dz=\int_0^{2\pi}\sqrt{2}\sin t\cdot 16\cos^2 t\cdot(-4)\sin t\, dt=\overbrace{-64\sqrt{2}}^{(-16)\cdot 4}\int_0^{2\pi}\sin^2 t\cos^2 t=-16\, I_2=-16\pi\sqrt{2}$$

$$C=\pi\sqrt{2}-16\pi\sqrt{2}=-15\pi\sqrt{2}$$

II način

pomoću Stoksove formule

površinski integral ↓

$$C=\int_C \vec v\, d\vec r = \iint_S \vec n\, \mathrm{rot}\,\vec v\, dS = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} dS$$

gdje je S površina koju zatvara kontura C, $\vec n = (\cos\alpha, \cos\beta, \cos\gamma)$ jedinični vektor normale na S

$$\mathrm{rot}\,\vec v = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & xy^2 & yz^2 \end{vmatrix} = (z^2-0)\vec i - (0-0)\vec j + (y^2-0)\vec k = (z^2, 0, y^2)$$

$$C = \iint_S (z^2\cos\alpha + y^2\cos\gamma)\, dS$$

posmatramo ravan $z=2x$ i vektor normale na ovu ravan, zato što je naša elipsa unutar ove ravni

$2x - z = 0 \qquad \vec n = (2, 0, -1)$

$|\vec n| = \sqrt{4+1} = \sqrt{5} \qquad \vec n_0 = \left(\frac{2}{\sqrt5}, 0, -\frac{1}{\sqrt5}\right)$

projekcija površi S na xOy ravan je elipsa $x^2+2y^2=4$

prema tome

$$C = \iint_S (z^2\cos\alpha + y^2\cos\gamma)\, dS = \iint_{D'} z^2\, dy\, dz - \iint_{D''} y^2\, dx\, dy$$

$x^2+2y^2=4$
$z=2x$

$\left(\frac{z}{2}\right)^2 + 2y^2 = 4 \quad /\cdot 4$

$z^2+8y^2=16 \quad /:16$

$\frac{z^2}{16}+\frac{y^2}{2}=1$

Projekcija površi S na yOz ravan je elipsa D': $\left\{ \frac{z^2}{16}+\frac{y^2}{2}=1 \right.$

$$\iint_{D'} z^2\, dy\, dz = \left| \begin{matrix} \frac{z}{4} = r\cos\varphi & z = 4r\cos\varphi & dy\,dz = 4\sqrt2\, r\,dr\,d\varphi \\ \frac{y}{\sqrt2} = r\sin\varphi & y = \sqrt2\, r\sin\varphi & \\ & 0\le\varphi\le 2\pi & \\ & 0\le r\le 1 & \end{matrix} \right| = \iint_{D_1'} 16 r^2\cos^2\varphi\, 4\sqrt2\, r\, dr\, d\varphi =$$

$$= 64\sqrt2 \int_0^1 r^3\, dr \int_0^{2\pi}\cos^2\varphi\, d\varphi = 64\sqrt2\int_0^1 r^3\, dr \int_0^{2\pi} \frac12(1+\cos2\varphi)\, d\varphi = 32\sqrt2 \cdot \frac{r^4}{4}\Big|_0^1 \left(2\pi - \frac12 \sin2\varphi\Big|_0^{2\pi}\right) =$$

$$= 16\sqrt2\,\pi$$

Projekcija površi S na xOy ravan je elipsa D'': $\left\{ \frac{x^2}{4}+\frac{y^2}{2}=1 \right.$

$$\iint_{D''} y^2\, dx\, dy = \left| \begin{matrix} \frac{x}{2} = r\cos\varphi & x = 2r\cos\varphi & dx\,dy = 2\sqrt2\, r\,dr\,d\varphi \\ \frac{y}{\sqrt2} = r\sin\varphi & y = \sqrt2\, r\sin\varphi & 0\le\varphi\le2\pi \\ & & 0\le r\le 1 \end{matrix} \right| = \int_0^1\int_0^{2\pi} 2r^2\sin^2\varphi \cdot 2\sqrt2\, r\, dr\, d\varphi =$$

$$= 4\sqrt2 \int_0^1 r^3\, dr \int_0^{2\pi} \sin^2\varphi\, d\varphi = \ldots = \pi\sqrt2$$

$$C = 16\pi\sqrt2 - \pi\sqrt2 = 15\pi\sqrt2$$

(#) Izračunati cirkulaciju polja $\vec{v} = x\vec{i} + y\vec{j} + (x+y-1)\vec{k}$ duž odsječka prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.

Rj. Cirkulacija vektorskog polja $\vec{v}$ ($\vec{v}=(V_x,V_y,V_z)$) duž krive c je integral

$$C = \int_c V_x dx + V_y dy + V_z dz .$$

U našem slučaju $\vec{v} = (x, y, x+y-1)$, dok je c dio prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.

Imamo krivolinijski integral druge vrste

$$C = \int_c x\, dx + y\, dy + (x+y-1)\, dz$$

$A(1,1,1)$
$B(2,3,4)$

Kako glasi jednačina prave kroz dvije tačke u prostoru?

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3} \quad (=t)$$

Napišimo pravu u parametarskom obliku:

$x = t+1$
$y = 2t+1$
$z = 3t+1$

Dio prave između tačke $A(1,1,1)$ i $B(2,3,4)$ je za $t \in [0,1]$.

$dx = dt, \quad dy = 2\,dt, \quad dz = 3\,dt$

$$C = \int_0^1 (t+1)\,dt + (2t+1)\,2\,dt + (3t+1)\,3\,dt = \int_0^1 (t+1+4t+2+9t+3)\,dt$$

$$= \int_0^1 (14t+6)\,dt = 14 \cdot \frac{1}{2} t^2 \Big|_0^1 + 6t \Big|_0^1 = 7+6 = 13$$

vrijednost cirkulacije polja

(#) Data su skalarna polja $f=xyz$, $g=xy+yz+zx$

a) Formirati vektorska polja $\vec{a}=\operatorname{grad} f$, $\vec{b}=\operatorname{grad} g$ i ispitati prirodu vektorskog polja $\vec{a}\times\vec{b}$.

b) Izračunati $\int_C (\vec{a}\times\vec{b})\cdot d\vec{r}$, gdje je C duž koja spaja tačke $O(0,0,0)$ i $B(1,2,3)$.

Rj.

a)
$$\operatorname{grad} f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

$$\left.\begin{aligned}\operatorname{grad} f&=(yz,\ xz,\ xy)\\ \operatorname{grad} g&=(y+z,\ x+z,\ x+y)\end{aligned}\right\} \Rightarrow \begin{aligned}\vec{a}&=(yz,xz,xy)\\ \vec{b}&=(y+z,x+z,x+y)\end{aligned}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ yz&xz&xy\\ y+z&x+z&x+y\end{vmatrix}=\left(x^2z+xyz-x^2y-xyz,\ -(xyz+y^2z-xy^2-xyz),\ xyz+yz^2-xyz-xz^2\right)=$$

$$=\left(x^2z-x^2y,\ xy^2-y^2z,\ yz^2-xz^2\right)$$

Priroda vektorskog polja - misli se da odredimo da li je polje potencijalno ili solenoidno ($\operatorname{rot}(\vec{a}\times\vec{b})=\vec{0}$ ili $\operatorname{div}(\vec{a}\times\vec{b})=0$).

$$\operatorname{rot}\vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ v_x&v_y&v_z\end{vmatrix}$$

$$\operatorname{rot}(\vec{a}\times\vec{b})=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x^2(z-y)&y^2(x-z)&z^2(y-x)\end{vmatrix}=\left(z^2-y^2,\ -(z^2-x^2),\ y^2-x^2\right)\neq\vec{0}$$

$$\underline{\operatorname{div}\vec{v} = \nabla\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}}$$

$$\operatorname{div}(\vec{a}\times\vec{b}) = 2x(z-y) + 2y(x-z) + 2z(y-x) =$$

$$= 2xz - 2xy + 2xy - 2yz + 2yz - 2xz = 0$$

$\vec{a}\times\vec{b}$ je solenoidno polje.

b)

$$\int_C (\vec{a}\times\vec{b})\, d\vec{r} = \left| \begin{array}{l} \vec{a}\times\vec{b} = (x^2(z-y),\ y^2(x-z),\ z^2(y-x)) \\ d\vec{r} = (dx,\ dy,\ dz) \end{array} \right|$$

$$= \int_C x^2(z-y)\,dx + y^2(x-z)\,dy + z^2(y-x)\,dz \overset{(*)}{=}$$

Određimo parametarski oblik duži OB $\quad O(\overset{x_1}{0},\overset{y_1}{0},\overset{z_1}{0})$ $\quad B(\overset{x_2}{1},\overset{y_2}{2},\overset{z_2}{3})$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \quad (=t)$$

$$OB: \begin{cases} x = t \\ y = 2t \\ z = 3t \\ 0 \le t \le 1 \end{cases} \qquad \begin{array}{l} dx = dt \\ dy = 2\,dt \\ dz = 3\,dt \end{array}$$

$$\overset{(*)}{=} \int_0^1 \left(t^2 \cdot t + 4t^2\cdot(-2t) + 9t^2\cdot t\right)dt =$$

$$= \int_0^1 (t^3 - 8t^3 + 9t^3)\,dt = \int_0^1 2t^3\,dt = 2\cdot\frac{t^4}{4}\Big|_0^1 = \frac{1}{2}$$ traženo rješenje

(#) Izračunati fluks vektorskog polja

$$\vec{v} = (x, -y^2, x^2 + z^2 - 1)$$

po unutrašnjoj strani sfere $x^2+y^2+z^2=1$.

Rj

Prisjetimo se

Fluks vektorskog polja se računa po formuli

$$\Phi = \iint_S v_x\,dy\,dz + v_y\,dx\,dz + V_z\,dx\,dy$$

U našem slučaju

$$\Phi = \iint_S x\,dy\,dz - y^2\,dx\,dz + (x^2+z^2-1)\,dx\,dy$$

gdje je S unutrašnja strana sfere sa centrom u $C(0,0,0)$ poluprečnika 1.

Ako iskoristimo formulu Gauss-Ostrogradski imamo

$$\Phi = \iiint_\Omega (1-2y+2z)\,dx\,dy\,dz =$$

| Ω je unutrašnjost te sfere - ako uvedemo sferne koordinate imamo
| $x=\rho\sin\varphi\cos\alpha$
| $y=\rho\sin\varphi\sin\alpha$
| $z=\rho\cos\varphi$
| $dx\,dy\,dz=\rho^2\sin\varphi\,d\rho\,d\varphi\,d\alpha$

$$\Omega \xrightarrow{\text{transformiše}} \Omega' : \begin{cases} 0 \le \rho \le 1 \\ 0 \le \varphi \le \pi \\ 0 \le \alpha \le 2\pi \end{cases}$$

$$= \iiint_{\Omega'} (1 - 2\rho\sin\varphi\sin\alpha + 2\rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\alpha =$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{1} \left(\rho^2 \sin\varphi - 2\rho^3 \sin^2\varphi \sin\theta + 2\rho^3 \sin\varphi \cos\varphi\right) d\rho$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \left(\frac{1}{3}\rho^3\Big|_0^1 \sin\varphi - 2\cdot\frac{1}{4}\rho^4\Big|_0^1 \sin^2\varphi \sin\theta + 2\cdot\frac{1}{4}\rho^4\Big|_0^1 \sin\varphi\cos\varphi\right) d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \left(\frac{1}{3}\sin\varphi - \frac{1}{4}(1-\cos 2\varphi)\sin\theta + \underbrace{\frac{1}{2}\sin\varphi\cos\varphi\right) d\varphi}_{-\frac{1}{2}\cos\varphi\, d(\cos\varphi)}$$

$$= \int_0^{2\pi} \left(-\frac{1}{3}\underbrace{\cos\varphi\Big|_0^{\pi}}_{-1-1} - \frac{1}{4}\sin\theta\left(\underbrace{\varphi\Big|_0^{\pi}}_{\pi} - \frac{1}{2}\underbrace{\sin 2\varphi\Big|_0^{\pi}}_{0-0}\right) - \frac{1}{2}\cdot\frac{1}{2}\underbrace{\cos^2\varphi\Big|_0^{\pi}}_{1-1=0}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{2}{3} - \frac{1}{4}\pi\sin\theta\right) d\theta = \frac{2}{3}\theta\Big|_0^{2\pi} + \frac{1}{4}\pi\underbrace{\cos\theta\Big|_0^{2\pi}}_{1-1} =$$

$$= \frac{4\pi}{3}$$

$$1 = \sin^2\varphi + \cos^2\varphi$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$

$$1 - \cos 2\varphi = 2\sin^2\varphi$$

$$\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi)$$